CH. 10
Mechanical Springs

1. Mechanical Springs:

A spring is a mechanical device which is used for the efficient storage and relase of energy and it recovers its shape when load is released. Depending upon the requirements, a spring can take different shapes,

1. Helical Compression or tension springs, in which the major stress is shear due to twisting. They are made of wire Coiled into a helical form, the load being applied a long the axis of the helix.

2. Helical torsion springs, in which the major stresses are tensile and compressive due to bending, the torque being applied to the axis of the helix.

3. leaf springs, in which the major stresses are tensile and compressive. They are composed of flat bars of varying lengths Clamped together so as to obtain greater efficiency and resilience. They may be full elliptic, Semi elliptic or Cantilever.

4. Belliville springs, in which the major stresses are tensile and Compressive are composed of Cone discs which may be stacked up to give a variety of spring load - deflection characteristics.

The main applications of the springs as follows:

(i) to act a reservoir of energy, e.g. springs in clocks, toys or movie - Cameras.

(ii) to absorb shocks and vibrations, e.g. vehicle suspension spring.

(iii) to return the mechanical part to its original position, when it has been temporarily displaced, e.g. springs in valves, clutches and linkages.

(iv) to measure force, e.g. spring balance.

Strength and flexibility are two essential requirements of spring - design. In this Chapter, the discussion is restricted to helical and leaf springs.

2. Helical Springs - Stress Equation.

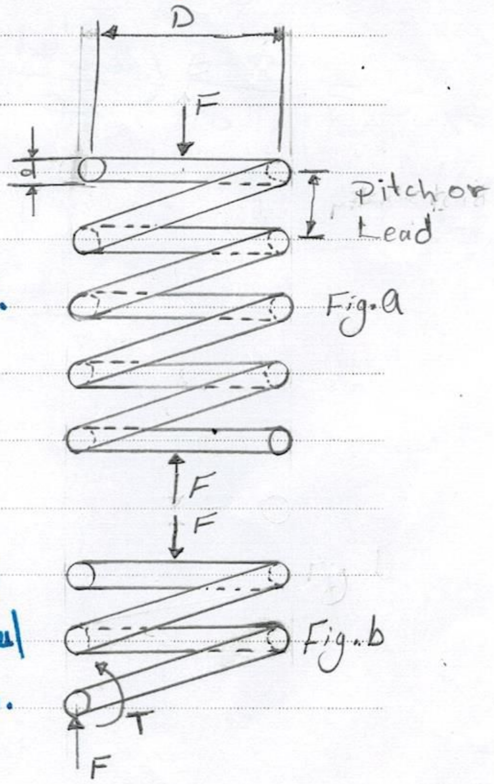


Fig. a. shows around - wire helical Compression spring loaded by the axial force F.

Where: D: mean coil diameter (mm).
d: wire diameter (mm).
F: axial force (N).

When the spring is subjected to axial load, the material of the spring (wire) must provide the resisting torsional moment ($T = F\,D/2$) as shown in Fig. b.

Due to this torque a shear stress setup with in the material of the wire.

$$\tau = \frac{T \cdot r}{J} \qquad (1)$$

The direct shear stress due to the axial load is,

$$\tau = \frac{F}{A} \qquad (2)$$

and is small compared with torsional shear stress in (equ. 1) and can be neglected.

The maximum stress in the wire may be computed by

$$\tau_{max} = \frac{T \cdot r}{J} + \frac{F}{A}$$

Replacing the terms by

$T = \frac{FD}{2}$, $r = \frac{d}{2}$, $J = \frac{\pi d^4}{32}$, and $A = \frac{\pi d^2}{4}$, gives

$$\tau_{max} = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}$$

∴ the spring index (c) is equal to $C = \frac{D}{d}$

$$\rightarrow \quad \tau_{max} = \frac{8FD}{\pi d^3}\left(1 + \frac{0.5}{C}\right)$$

where: $K_s = 1 + \frac{0.5}{C} \cong$ shear-stress correction factor $= \frac{2C+1}{2C}$

$$\therefore \tau_{max} = K_s \cdot \frac{8FD}{\pi d^3} \qquad (3)$$

For most springs, C will range from about 6 to 12. Equation (3) is quite general and applies for both static and dynamic loads.

- The Curvature Effect

When the bar is bent in the form of a helical coil, the length of the inside fibre is less than the length of the outside fibre. This results in stress concentration at the inside fibre of the coil. Equation (3) does not take into consideration the effect of stress concentration due to the curvature of the coil. The equation for resultant stress, which includes direct shear stress, torsional shear stress and stress concentration due to curvature, was derived by A. M. Wahl present the following equations.

$$k_w = \frac{4C-1}{4C-4} + \frac{0.615}{C} \qquad (4)$$

Where K_w is called the Wahl factor.

and

$$k_B = \frac{4C+2}{4C-3} \qquad (5)$$

Where k_B is called Bergsträsser factor.

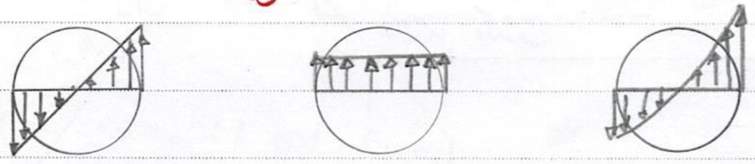
Since, the results of two equations differ by less than 1%, then, Eq. (5) is preferred.

The superimposition of torsional shear stress, direct shear stress and curvature effect is shown in fig. below. The Wahl correction factor K_w consists of two factors, correction factor K_s for direct shear stress and Correction factor K_c for the curvature effect, Therefor,

$$K_B = K_S \cdot K_C$$

Or,

$$K_C = \frac{K_B}{K_S} \qquad (6)$$



pure torsional stress. Direct shear stress. Combined torsional, direct and curvature shear stresses.

The values of these three factors are given in table below. The stress Concentration due to curvature is localised at the inner side of the spring. When the spring is subjected to a static force, the effect of stress Concentration is neglected due to localised yielding. The load-stress equation for static forces, therefor, is given by equation (3). When the spring is subjected to fluctuating forces, the endurance strength is reduced due to stress Concentration and K_c is used as the stress Concentration factor.

In this book we will use K_B to predict the largest shear stress.

Table explain the spring stress factors.

C	Kw	Ks	Kc
5	1.311	1.100	1.192
5.5	1.278	1.091	1.171
6	1.253	1.083	1.157
6.5	1.231	1.077	1.143
7	1.213	1.071	1.133
7.5	1.197	1.067	1.122
8	1.184	1.063	1.114
8.5	1.172	1.059	1.107
9	1.162	1.056	1.100
10	1.145	1.050	1.090
11	1.131	1.045	1.082
12	1.119	1.042	1.074

Note:

For square wire and rectangular wire springs $C = D/t$, when (D) mean coil diameter and (t) radial thickness of the wire.

3. Deflection of Helical Springs.

$$y = \frac{8 F D^3 N}{d^4 G}$$

where:

y: deflection.
N: number of active Coils.
G: Shear modulus.

The stiffness of the spring or the spring rate (K) is the function of the geometrical dimensions of the springs and the material of the spring.

$$K = \frac{F}{y} = \frac{d^4 G}{8 D^3 N} = \frac{d G}{8 C^3 N}$$

where $N = N_a$ = number of active coils.

The above equations Can be used for extension and compression springs.

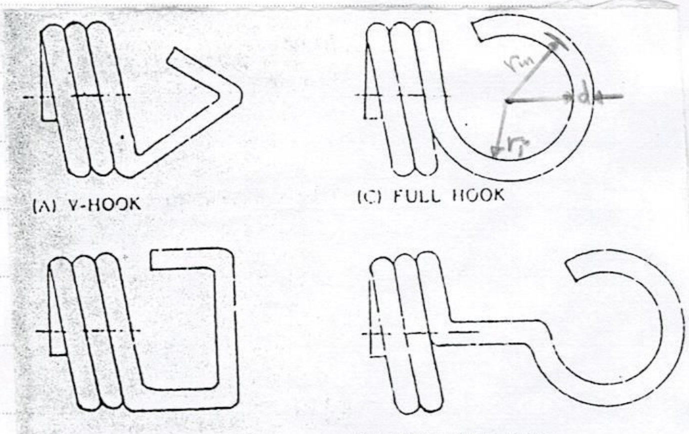
4. Ends of Helical Springs.

A. Extension Springs.

These springs must have some means of applying the load. Types of ends are shown in Fig. below. The end should be designed in such a way that stress Concentration at the bend is minimum.

Sometimes the effect of stress concentration in springs is so severe that the spring body becomes stronger than the end and failure occurs in the end coils.

For helical extension springs, all coils are active. The number of active coils (N) is the same as the total number of coils (N_t).



(A) V-HOOK (C) FULL HOOK (B) RECTANGULAR HOOK (D) EXTENDED HOOK

The stress concentration factor at these ends is,

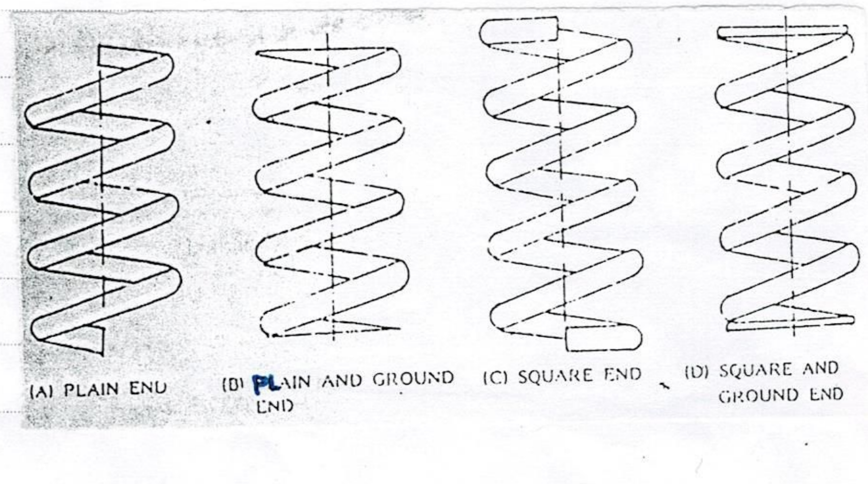
$$K_t = \frac{r_m}{r_i}$$

* show the book.

which hold for bending and torsional stress.

B. Compression Springs.

Type of ends for compression springs are shown in the figure.



(A) PLAIN END (B) PLAIN AND GROUND END (C) SQUARE END (D) SQUARE AND GROUND END

The end types used results in dead or in active turns at each end of the spring, so to obtain the number of active turns.

$$N = N_t - N_e$$

Where:

N: no. of active turns.

N_t: total no. of turns. (Calculated).

N_e: no. of inactive turns.

- $N_e = 0$ For plain ends.
- $N_e = 1$ For plain and ground.
- $N_e = 2$ For squared or Closed.
- $N_e = 2$ For squared and ground.

5. Spring Materials.

Spring materials are illustrate with their uses in table 10-3.

To find the Ultimate strength for any spring material.

$$\sigma_{ut} = \frac{A}{d^m}$$

Where:

d: wire diameter.

A & m: are Constant presented in table 10.4

Note: $\sigma_y = 0.75\ \sigma_{ut}$.

$\tau_y = 0.577\ \sigma_y$.

$\therefore \tau_y = 0.577(0.75)\sigma_{ut} = 0.433\ S_{ut} \simeq 0.45\ S_{ut}$

Example: A helical spring whose mean diameter of coils is (8) times that of the wire is to absorb 400N.m of energy. The initial compression of the spring is (50 mm) and compresses by additional (70mm) while absorbing the shock. The maximum allowable stress is 400 Mpa and G = 84 Gpa. Determine the diameter of the wire and the number of active turns. Neglect the effect of stress concentration.

Solution:

$K = \frac{F}{y}$ N/mm,

∴ the initial load on the spring is (50K).

As the spring further compresses by (70mm), the maximum load on the spring is equal to.

$F = (50 + 70)K = 120\ K$ (N).

⇒ Mean spring force (during compression) $= \frac{120K + 50K}{2}$

$= 85\ K$ (N).

The energy absorbed during shock $= 85K * 70$

$= 5950\ K$ (N.mm).

∴ $400 * 10^{6} = 5950 K * 10^{-3} \Rightarrow K = 67.22$ MN/m.

Max. spring force $= 67.22 * 10^{6} * 120 * 10^{-3} = 8.07$ MN.

∵ $\tau = K_s \cdot \frac{8FD}{\pi d^3} \Rightarrow d^3 = \frac{K_s \cdot 8 \cdot F \cdot D}{\pi \cdot \tau} \Rightarrow d^2 = \frac{K_s \cdot 8 \cdot F \cdot C}{\pi \cdot \tau}$

∴ $d^2 = \frac{(1 + \frac{0.5}{8}) * 8 * 8.07 * 10^{6} * 8}{\pi \cdot 400 * 10^{6}} = 0.43 \Rightarrow d = 0.66\ m.$

$= 660\ mm.$

$\therefore$ Mean diameter $D = C * d$

$= 8 * 660 = 5280$ mm

$= 5.28$ m.

$\because K = \dfrac{d\,G}{8\,C^3 N_a}$

$\Rightarrow N_a = \dfrac{8 \cdot C^3 \cdot K}{d \cdot G}$

$= \dfrac{8 * 8^3 * 67.22 * 10^6}{0.66 * 84 * 10^9}$

$= 4.96$ turns.

9. Helical Compression Spring Design for Static Service.

To maintain linearity when a spring is about to close, it is necessary to avoid the gradual touching of coils. A helical coil spring force-deflection characteristic is ideally linear. Practically, it is nearly so, but not at each end of the force-deflection curve. The spring force is not reproducible for very small deflections, and near closure, nonlinear behavior begins as the number of active turns diminishes as coils begin to touch. The designer confines the spring's operating point to the central 75 percent of the curve between no loud, $F=0$, and closur, $F=F_s$. Thus, the maximum operating force should be limited to $F_{max} \leq \frac{7}{8} F_s$. Defining the fractional overrun to closure as ξ, where

$$F_s = (1+\xi) F_{max}$$
$$= (1+\xi)\left(\frac{7}{8}\right) F_s$$

From the outer equality $\xi = 1/7 = 0.143 \cong 0.15$, Thus it recommended that $\xi \geq 0.15$

In addition to the relationships and material properties for springs, we now have some recommended design conditions to follow, namely:

$$4 \leq C \leq 12$$
$$3 \leq N_a \leq 15$$

$$\xi \geq 0.15$$

$$n_s \geq 1.2$$

where n_s is the factor of safety at closure (solid height).

When considering designing a spring for high volume production, the figure of merit can be the cost of the wire from which the spring is wound. The fom would be proportional to the relative material cost, weight density, and volume.

$$fom = -\,(\text{relative material cost})\,\frac{\gamma \pi^2 d^2 N_t D}{4}$$

For comparisons between steels, the specific weight γ can be omitted.

Spring design is an open-ended process. There are many decisions to be made, and many possible solution paths as well as solutions.

Static spring design

choose d

Over-a-rod

As-wound or set

$D = D_{rod} + d + \text{allow}$

Free

As-wound

$S_{sy} = \text{Const}(A)/d^m$ +

$$C = \frac{2\alpha-\beta}{4\beta} + \sqrt{\left(\frac{2\alpha-\beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}}$$

$$\alpha = \frac{S_{sy}}{n_s}, \quad \beta = \frac{8(1+\xi)F_{max}}{\pi d^2}$$

$$D = Cd$$

Set removed

$S_{sy} = 0.65\,A/d^m$

$$D = \frac{S_{sy}\pi d^3}{8 n_s (1+\xi) F_{max}}$$

In-a-hole

As-wound or set

$D = d_{hole} - d - \text{allow}$

+ Const is found from table 10-13.

where:

$$C = D/d$$

$$K_B = (4C+2)/(4C-3)$$

$$n_s = S_{sy}/\tau_s$$

$$OD = D+d$$

$$ID = D-d$$

$$N_a = G d^4 y_{max}/(8 D^3 F_{max})$$

N_t: table 10-2

L_s: table 10-2

$$L_0 = L_s + y_s$$

$$L_0)_{cr} = 2.63\, D/\alpha$$

$$fom = -(rel.cost)\, 8\pi^2 d^2 N_t D/4$$

* Design Strategy.

Make the a priori decisions, with hard-drawn steel wire the first choice relative material cost is 1. Chosse a wire size d. with all decisions made, generate a column of parameters: d, D, C, OD or ID, Na, Ls, Lo, Lo)cr, ns and fom. By incrementing wire sizes available, we can scan the table of parameters and apply the design recommendations by inspection. After wire sizes are eliminated, choose the spring design with the highest figure of merit. This will give the optimal design despite the presence of a discrete design veriable d and aggregation of equality and inequality Constraints.

Add the information of pg 512

8. Critical Frequency of Helical Springs.

The governing equation for a spring placed between two flat and parallel plates is the classical wave equation and is

$$\frac{\partial^2 u}{\partial^2 y} = \frac{m}{k l^2} \cdot \frac{\partial^2 u}{\partial^2 t}$$

where: k = spring rate (stiffness).
l = total length of spring wire between plates.
m = mass of spring.

But $m = A l \rho = \frac{\pi d^2}{4} (\pi D N_a) \rho = \frac{\pi^2 d^2 D N_a \rho}{4}$

where $\rho = \frac{mass}{Volum}$ = mass density.

y = Coordinate along the axis of the spring.
u = motion of any partical at distance y.

The solution of this D.E yields.

$W_n = \frac{1}{2}\left(\frac{k}{m}\right)^{1/2}$ ≡ Natural frequency (rad/s) for a spring placed between two parallel flat plates.

$W_n = \frac{1}{4}\left(\frac{k}{m}\right)^{1/2}$ ≡ Natural frequency (rad/s) for a spring with one end aginst a flate plate and other end free.

The natural frequency should be from (15 to 20) times the frequency of the force or motion of the spring in order to avoid resonance.

If the frequency is not high enough, the spring should be re designed to increase the spring rate or decrease the mass.

$$\omega_n \geq (15 \rightarrow 20) * \omega_{\text{force or motion}}.$$

Example: An automotive single-plate clutch, with two pairs of friction surfaces, transmits a 300 N.m torque at 1500 r.p.m. The inner and outer diameters of the friction disck are 170 and 270 mm respectively. The cofficient of friction is 0.35. The normal force on the friction surfaces is exerted by nine helical Compression springs, so that the clutch is always engaged. The clutch is disengaged when the external force further Compresses the springs. The spring index is (5) and the number of active Coile are (6). The springs are made of patented and Cold-drawn steel wires of minimum tensile strength (MTS) (N/mm^2) as

Wire dia. d(mm)	1	1.2	1.4	1.6	1.8	2	2.5	3	3.6	4	4.5	5	6	7
MTS (N/mm^2)	1900	1860	1820	1780	1750	1720	1640	1570	1510	1480	1440	1390	1320	1260

The permissible shear stress for the spring wire is 30% of the ultimate tensile strength, and ($G = 81370\ N/mm^2$). Design the springs and specify their dimensions.

Use this relation to obtaine to the total normal force required to transmit the torque. $F = \frac{4T}{\mu(D+d)}$, where D is outer diameter and d is inner diameter for the disk. (Assume No Curvature effect).

10. Fatigue Loading.

In many applications, the force acting on the spring is not constant but varies in magnitude with time.

Let us consider a spring subjected to a fluctuating force, which changes its magnitude from F_{max} to F_{min} in the load cycle. The mean force F_m and the force amplitude F_a are given by.

$$F_a = \frac{F_{max} - F_{min}}{2} \quad \& \quad F_m = \frac{F_{max} + F_{min}}{2}$$

From Equation (3), the mean torsional shear stress is,

$$\tau_m = K_B \cdot \frac{8 \cdot F_m \cdot D}{\pi \cdot d^3}$$

where : K_B: is the Bergstrasser factor.

For torsional stress amplitude τ_a it is neccessary there fore;

$$\tau_a = K_B \frac{8 \cdot F_a \cdot D}{\pi \cdot d^3}$$

The corresponding endurance strength components for infinite life we found in ~~table 10-14~~.
Eqs' pg 298

To Calculated the design factor or factor of safety used this relation,

$$\frac{\tau_a}{\left(\frac{S_{sy}}{n}\right) - \tau_m} = \frac{\frac{1}{2} S_{se}'}{S_{sy} - \frac{1}{2} S_{se}'}$$

where : S_{sy} : the torsional yield strength.
S_{se}' : endurance limit in shear.
n : design factor.

* For Cold-drawn steel wires.

$S_{se}' = 0.21\ S_{ut}$

$S_{sy} = 0.42\ S_{ut}$

* For oil-hardened tempered steel wire.

$S_{se}' = 0.22\ S_{ut}$

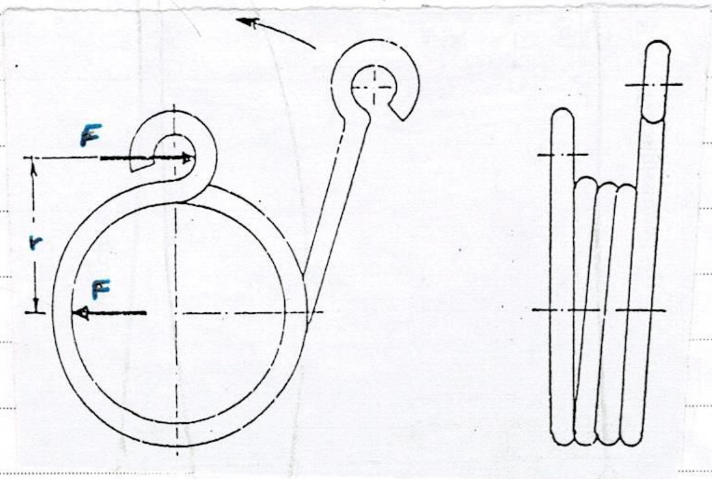
$S_{sy} = 0.45\ S_{ut}$.

Example: Aspring subjected to a load varying from 400 N to 1000 N is to be made of oil tempered, cold wound wire. Determine the diameter of the wire and the mean diameter of the coil for a design. factor of 1.25 based on Wahl's line. The spring index is to be at least (5). The free length of the spring should lie between (100 to 150 mm) and ($C = 5.5$). and ($S_{ut} = 1430\ N/mm^2$)

Solution:

11. Helical Torsion Springs.

A helical torsion spring is a device used to a particular component of a machine or mechanism. It is widely used in door-hinges, brush-holders, automobile-starters and door-locks. As shown in the figure below.



Using the curved-beam theory, the spring bending stresses are given by.

$$\sigma_b = K\left(\frac{M \cdot y}{I}\right)$$

where K is the stress concentration factor. For a wire of circular cross-section,

$$y = \frac{d}{2}, \quad I = \frac{\pi d^4}{64}, \quad M = F \cdot r$$

$\Rightarrow \sigma_b = K_i\,(32Fr/\pi d^3)$ for circular wire.

$\sigma_b = K_i\,(6Fr/a^3)$ for squar wire.

The expression for stress Concentration factor K was analytically derived by Wahl in to inner and outer fibers of Coil.

$$K_i = \frac{4C^2 - C - 1}{4C(C-1)} \quad ; \quad K_o = \frac{4C^2 + C - 1}{4C(C+1)}$$

The strain energy stored in the spring is given by

- $U = \frac{\sigma_b^2}{8E} *$ volume of the spring.

$\therefore \quad U = \frac{F^2 r^2 (\pi D N_b)}{2EI}$ for round wire.

- $U = \frac{\sigma_b^2}{6E} *$ volume of the spring.

$\therefore \quad U =$ for square wire.

The deflection in the direction of the force F is approximately $(r * \theta)$.

$\Rightarrow \quad r * \theta = \frac{\partial U}{\partial F}$

$\therefore \quad \theta = \frac{64 F r D N_b}{E d^4}$ for round wire.

$\theta =$ for squar wire.

The stiffness of the helical torsion spring is defined as the bending moment required to produce unit angulor displacement. Therefore.

$$K = \frac{F \cdot r}{\theta}$$

$$= \frac{E d^4}{64\, D N_a} \quad \text{for round wire.}$$

and $$K = \frac{F \cdot r}{\theta}$$

$$=$$

for square wire.

Example: A flexible shaft transmits (2.5 N.m) torque. It consists of a torsion spring in which the permissible stress is limited to (450 Mpa). what must be the diameter of the wire? Use (C) the index spring is (12), and from the following gauges in the table select the SWG.

SWG	4	5	6	7	8	9	10	11	12
dia mm	5.89	5.38	4.87	4.47	4.06	3.65	3.25	2.94	2.64

SHEET

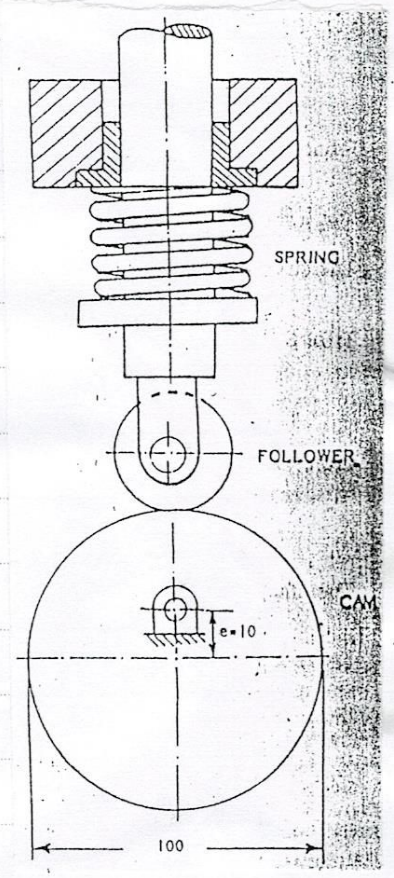
1. The table below gives particulars of Concentric helical springs, If the spring is subjected to an axial load of 400 N. determine for each spring. (G = 84 Gpa).

   a. the change in length.
   b. the amount of load Carried.
   c. the shear stress induced in the wire,

	Mean Coil dia	size of wire	Dia. of wire	No. of turns	free length
Inner spring	80 mm	8 SWG	4.064 mm	8	75 mm
outer spring	40 mm	6 SWG	4.877 mm	10	90 mm.

2. An eccentric Cam, 100 mm in diameter rotates with an eccentricity of 10 mm as shown in the figure. The roller follower is held against the Cam by means of a helical-Compression spring. The force between the Cam and the follower varies from 100 N at the lowest position to 350 N at the highest position of the follower. Design the spring from static Consideration and determine the factor of safety for fatigue Considerations. Neglect the effect of inertia forces.

Note: assume for this state the oil-hardened and tempered steel-wire SW grade, and $c = 6$.

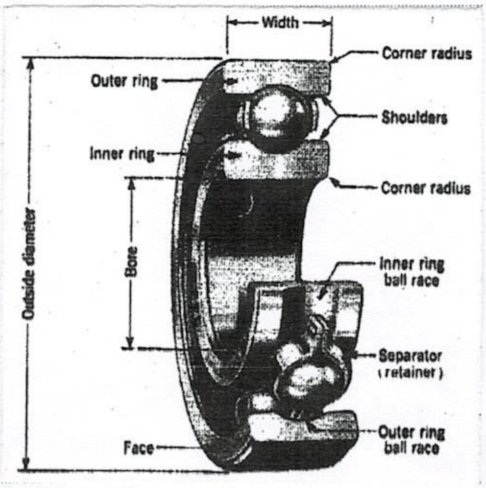


3. Design a Cantilever leaf spring to absorb (800 N m) of energy without exceeding a deflection of (150 mm) and the permissible stress (875 Mpa). The length of the spring is (600 mm) and the modulus of elasticity is (210 Gpa).

4. A 1000 mm long Cantilever Spring is composed of 8 graduated leaves and 1 extra full length leaf. The leaves are (50 mm) wide. A load of 2500 N at the end of the spring causes a deflection of 80 mm. determine the thickness of the leaves and the maximum bending stress in the full length leaves assuming first that the extra full length leaf has been prestressed to give the same stress in all the leaves, and then determine the stress in the full length extra leaf assuming no pre-stress. Take E = 210 Gpa.

5. A semi-elliptic multi-leaf spring is used for the suspension of the rear axle of a truck. It consists of two extra full-length leaves and ten graduated-length leaves including the master leaf. The centre-to-centre distance between the spring eyes is 1.2 m. The leaves are made of steel 55Si2Mo90 (S_{yt} = 1500 N/mm^2 and E = 207 kN/mm^2) and the factor of safety is 2.5. The spring is to be designed for a maximum force of 30 kN. The leaves are pre-stressed so as to equalize stresses in all leaves. Determin t, b, and δ.

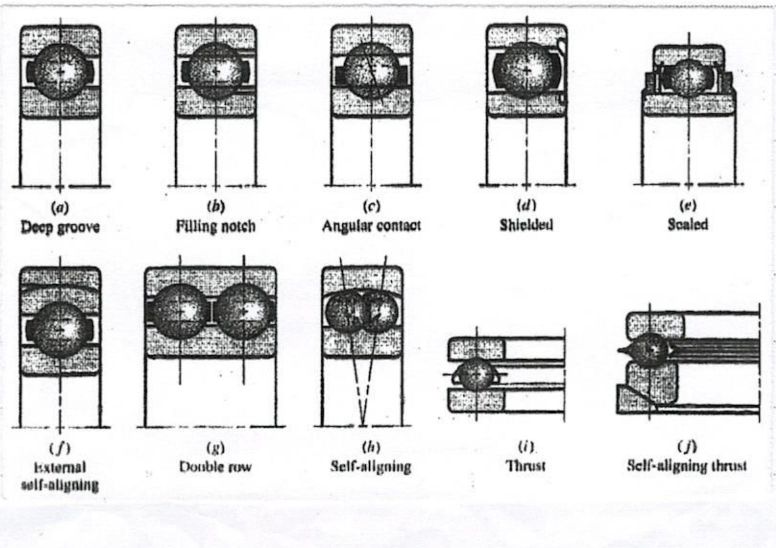
CH. 11
Rolling Contact Bearings.



The term rolling contact bearings encompasses the wide variety of bearings that use spherical balls or some type of roller between the stationary and moving elements, as shown in the figure. The most common type of bearing supports a rotating shaft resisting a combination of radial and axial (or thrust) loads. Some bearings are designed to carry only radial loads or only thrust loads.

11.1 Bearing Types

Some of the various types of standardized ball bearings which are manufactured are shown in the figure below.



11.2 Bearing Load - Life and Selection.

Rolling element bearings are generally standard items and can be purchased from specialist manufacturers. The selection of a bearing from manufacturers catalogue involves consideration of bearing load carrying capacity and the bearing geometry.

The basic dynamic load rating **C** is the constant radial load which a bearing can endure for 10^6 revolution.

The life of the bearing **L** is the number of revolution (or hours at some constant speed) which the bearing runs before the development of fatigue in any of the bearing components. Where the units of L are revolutions.

11.3 Book

$$L = \left(\frac{C}{F}\right)^a \quad \text{million revolution.}$$

Where: $F \cong$ the load.
$a \cong$ for ball bearings is (3) and (3.33) for roller bearings (cylinderical and tapered roller).

$$\therefore \quad C = F\,(L * 10^{-6})^{1/a}$$

Many bearing manufacturers publish ratings for bearings corresponding to a particular number of hours life at a specified rotational speed.

The designer task is to determine which value of Catalogue rating to use given a set of particular values for P_d, L_d and n_d : $FL^{1/a} = \text{constant} \Rightarrow F_1 L_1^{1/a} = F_2 L_2^{1/a} \Rightarrow C_{10} L_{10}^{1/a} = F L^{1/a}$

$$C_{10} = F_D \left(\frac{L_D \cdot n_D \cdot 60}{L_R \cdot n_R \cdot 60} \right)^{1/a}$$

Where:

C_{10} = Catalogue radial rating (N, kN)

F_D = required radial design load (N, kN)

L_D = required design life (rev, hours)

n_D = required design speed (rpm)

L_R = Catalogue rated life (rev, hours)

n_R = Catalogue rated speed (rpm)

Example: A straight cylindrical roller bearing operates with a load of 7.5 kN. The required life 8760 hours. at 1000 rpm. What load rating should be used for selection from the catalogue?

Solution:

$$C = F\,(L * 10^{-6})^{1/a}$$

$$= 7500\,(8760 * 1000 * 60 * 10^{-6})^{1/3.33}$$

$$= 49.2\ kN.$$

Example: A catalogue lists the basic dynamic load rating for a ball bearing to be 33800N for a rated life of one million revolutions. What would be the expected L_{10} life of the bearing if it were subjected to 15000N and determine the life in hours that this corresponds to if the speed of rotation is 2000 rpm.

Solution:

$\because L = \left(\frac{C}{F}\right)^a$ million revolution.

$\because a = 3$ (ball)

$$\therefore L = 10^6\left(\frac{33800}{15000}\right)^3 = 11.44 * 10^6 \text{ rev}$$

$$= L_{10} \text{ life at } 15000N.$$

If the rotational speed is 2000 rpm

$\Rightarrow L = 11.44 * 10^6 / (2000 * 60) = 95$ hours operation.

This is not very long and illustrates the need to use a bearing with a high basic dynamic load rating.

Example:

A single-row deep groove ball bearing is required to Carry a radial load of 2.8 kN and provide axial location for a shaft of 30 mm diameter rotating at 1500 rpm. An L_{10} life of 10000 hours is required select an appropriate bearing?

Solution:

$\therefore \; L = \left(\frac{C}{F}\right)^{a}$ million revolutions.

$\Rightarrow \; C = F \, L^{1/a}$

$a = 3$ (ball).

The total No. of revolution in life is

$10000 * 1500 * 60 = 900$ million.

$\Rightarrow \; L = 900$

The load is purely radial so,

$F = 2800$ N.

$\therefore \; C = F \, L^{1/3} = 2800 * 900^{1/3}$

$= 27033$ N

Reference to the deep groove bearing chart table 11.3, shows a suitable bearing Could be ...

11.3 Bearing Survival: Reliability versus Life

At constant load, the life measure distribution of rolling-contact bearings is right skewed. Unlike the development of normal distribution, we will begin with the definition of the reliability, R, for a Weibull distribution of the life measure, x. The Weibull is by far the most pupular, largely because of its ability to adjust to varying amount of skewness. If the life measure is expressed in dimensionless form as $x = L/L_{10}$, then the reliability can be expressed as

$$R = \exp\left[-\left(\frac{x - x_0}{\theta - x_0}\right)^b\right] \qquad x \geq x_0 \geq 0 \qquad (1)$$

Where R = reliability

x = life measure dimensionless variate, L/L_{10}

x_0 = guaranteed, or "minimum" value of x

θ = characteristic parameter, or scale value ($\theta \geq x_0$). For rolling-contact bearings, this corresponds to the 63.2121 percentile value of x

b = shape parameter that controls the skewness, ($b > 0$). For rolling-contact bearings, $b \cong 1.5$

From the derivative of equation (1), the Weibull probability density function $f(x)$, is given by

$$f(x) = \begin{cases} \dfrac{b}{\theta - x_0}\left(\dfrac{x - x_0}{\theta - x_0}\right)^{b-1} \exp\left[-\left(\dfrac{x - x_0}{\theta - x_0}\right)^b\right] & x \geq x_0 \geq 0 \\ 0 & x < x_0 \end{cases}$$

The mean and standard deviation of $f(x)$ are

$$\mu_x = x_0 + (\theta - x_0)\,\Gamma\left(1 + \frac{1}{b}\right)$$

$$\hat{\tilde{\sigma}}_x = (\theta - x_0)\left(\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right)\right)^{1/2}$$

Where Γ is the gamma function, and is found tabulated in Table A-32.

Given a specific required reliability, solving Eq.(1) for x yields

$$x = x_0 + (\theta - x_0)\left(\ln \frac{1}{R}\right)^{1/b}$$

The coefficient of variation of the dimensionless life is

$$C_x = \frac{\hat{\tilde{\sigma}}_x}{\mu_x}$$

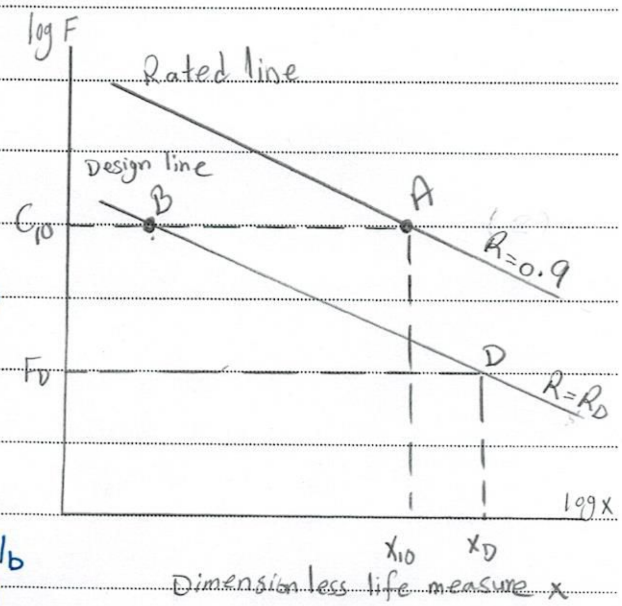
11.4 Relating Load, Life, and Reliability

This is the designer's problem. The desired load is not the manufacturer's test load or catalog entry. The desired speed is different from the vendor's test speed, and the reliability expectation is typically much higher than the 0.9 accompanying the catalog entry. Figure below shows the situation. The catalog information is plotted as point A, whose coordinates are (the logs of) C_{10} and $x_{10} = L10/L_{10} = 1$, a point on the 0.9 reliability contour. The design point is at D, with the coordinates (the logs of) F_D and x_D, a point that is on the $R = R_D$ reliability contour. The designer must move from point D to point A via point B as follows. Along a constant reliability contour (BD), recall Eq.

$$F_B X_B^{1/a} = F_D X_D^{1/a}$$

from which

$$F_B = F_D \left(\frac{X_D}{X_B}\right)^{1/a} \quad (2)$$



Along a constant load line (AB), Eq. (1) applies

$$R_D = \exp\left[-\left(\frac{X_B - X_0}{\theta - X_0}\right)^b\right]$$

solving for X_B gives

$$X_B = X_0 + (\theta - X_0)\left(\ln \frac{1}{R_D}\right)^{1/b}$$

Now substitute this in Eq. (2) to obtain

$$F_B = F_D\left[\frac{X_D}{X_0 + (\theta - X_0)(\ln 1/R_D)^{1/b}}\right]^{1/a}$$

However, $F_B = C_{10}$, and including an application factor a_f with the design load,

$$C_{10} = a_f F_D\left[\frac{X_D}{X_0 + (\theta - X_0)[\ln(1/R_D)]^{1/b}}\right]^{1/a} \quad (3)$$

Eq. (3) can be simplified slightly for calculator entry by noting that

$$\ln\frac{1}{R_D} = \ln\frac{1}{1-P_f} = \ln(1 + P_f + \cdots) \simeq P_f = 1 - R_D$$

Where P_f is the probability for failure, Eq. (3) can be written as

$$C_{10} \simeq a_f F_D\left[\frac{X_D}{X_0 + (\theta - X_0)(1 - R_D)^{1/b}}\right]^{1/a} \quad R > 0.9 \quad (4)$$

Either Eq.(2) or Eq.(3) may be used to convert from a design situation with a desired load, life, and reliability to catalog load rating based on a rating life at 90 percent reliability.

Note: when $R_D = 0.9$, the denominator is equal to one, and the equation reduces to

$$C_{10} = F_D \left(\frac{L_D}{L_R}\right)^{1/a} = F_D \left(\frac{L_D n_D 60}{L_R n_R 60}\right)^{1/a}$$

Eq.(3) can be solved for the reliability R_D in terms of C_{10}, the basic load rating of selected bearing:

$$R = \exp\left(-\left\{\frac{x_D\left(\frac{a_f F_D}{C_{10}}\right)^a - x_0}{\theta - x_0}\right\}^b\right)$$

Eq.(4) can likewise be solved for R_D:

$$R \simeq 1 - \left\{\frac{x_D\left(\frac{a_f F_D}{C_{10}}\right)^a - x_0}{\theta - x_0}\right\}^b \qquad R \geq 0.9$$

11.5 Combined Radial and Thrust Loading

When both radial and thrust loads are exerted on a bearing:

$$F_e = X_i V F_r + Y_i F_a$$

Where F_e = equivalent radial load

X = radial factor (given in bearing catalogues)

$$i = \begin{cases} 1 & F_a/(VF_r) \leq e \\ 2 & F_a/(VF_r) > e \end{cases}$$

$$V = \begin{cases} 1 & \text{for either ring rotates} \\ 1.2 & \text{for outer ring rotates} \end{cases}$$

F_r = applied radial load

Y = thrust factor (given in bearing catalogues).
F_a = applied thrust load.

Table 1 : Equivalent Radial load Factors for Ball Bearings.

F_a/C_o	e	X_1	Y_1	X_2	Y_2
0.014*	0.19	1	0	0.56	2.30
0.021	0.21	1	0	0.56	2.15
0.028	0.22	1	0	0.56	1.99
0.042	0.24	1	0	0.56	1.85
0.056	0.26	1	0	0.56	1.71
0.070	0.27	1	0	0.56	1.63
0.084	0.28	1	0	0.56	1.55
0.110	0.30	1	0	0.56	1.45
0.17	0.34	1	0	0.56	1.31
0.28	0.38	1	0	0.56	1.15
0.42	0.42	1	0	0.56	1.04
0.56	0.44	1	0	0.56	1.00

* Use 0.014 if $F_a/C_o < 0.014$.

Note: The static load rating comes from the equitions

$C_o = M\, n_b\, d_b^2$ (ball bearings)

$C_o = M\, n_r\, l_c\, d$ (roller bearings)

Where: n_b = number of balls , n_r = number of rollers

d_b = diameter of balls (mm) , d = diameter of rollers (mm)

l_c = length of contact line (mm)

M = see table pg 563

SHEET NO.

① A ball bearing for an industrial grinder is chosen to with stand a radial load of 1300 N and have an L_{10} life of 3600 hours at 3000 rpm. The manufacturers Catalogue rating is based on an L_{10} life of 3800 hours at 1000 rpm. What load should be used to enter into the catalogue for selection?

② Consider SKF, which rates its bearings for one million revolutions, so that L_{10} life is $L_{cat}\, n_{cat} = 10^6$ revolutions. The $L_d\, n_d$ produces a familiar number. If you desire a life of 5000 hours at 1725 rpm with a load of 400 Ibf with a reliability 90 percent, for which Catalog rating would you search in an SKF catalog?

③ An SKF 6210 angular-contact ball bearing has an axial load of 1.8 kN applied, a radial load of 2.2 kN applied with the outer ring stationary. The basic static load rating C_0 is 19.8 kN. Estimate the L_d life at a speed of 720 rpm? If the basic load rating C_{10} is 35 kN.

CH. 12
Lubrication and Journal Bearing

The object of lubrication is to reduce friction, wear, and heating of machine parts which move relative to each other. A lubrication is any substance which, when inserted between the moving surfaces, accomplishes these purposes. In a sleeve bearing, a shaft, or journal, rotates or oscillates within a sleeve, or bearing, and the relative motion is sliding. In an antifriction bearing, the main relative motion is rolling. A follower may either roll or slide on the cam. Gear teeth mate with each other by a combination of rolling and sliding. Pistons slide within their cylinders. All these applications require lubrication to reduce friction, wear, and heating.

12.1 Types of Lubrication.

1. Hydrodynamic lubrication means that the load-carrying surfaces of the bearing are separated by a relatively thick film of lubricant, so as to prevent metal-to-metal contact. Hydrodynamic lubrication does not depend up on the introduction of lubricant under pressure, though that may occur; but it does require the existence of an adequate supply at all times. The film pressure is created by the moving surface itself pulling the lubricant into a wedge-shaped zone at a velocity sufficiently high to create the pressure

necessary to separate the surfaces aginst the load on the bearing. Hydrodynamic lubrication is also called full-film, or fluid lubrication.

2. Hydrostatic lubrication is obtained by introduction the lubricant, which is some times air or water, into the load-bearing area at apressure high enough to separate the surfaces with a relatively thick film of lubricant.

3. Elastohydrodynamic lubrication is the phenomenon that occurs when a lubricant is introduced between surfaces which are in rolling contact, such as mating gears or rolling bearing.

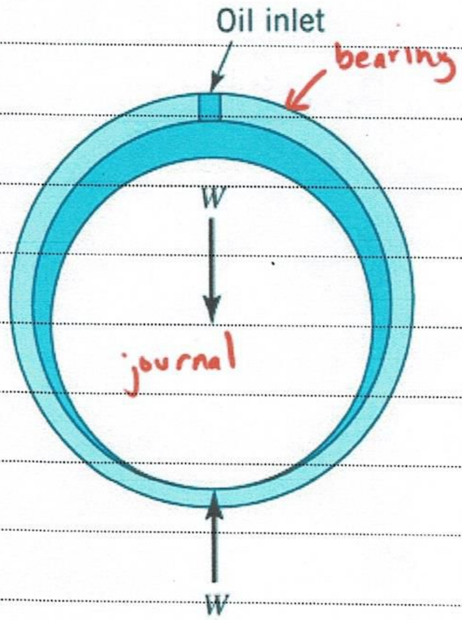
4. Many effects may prevent the buildup of a film thick enough for full-film lubrication. When this happens, the highest asperities may be separated by lubricant film only several molecular dimensions in thickness. This is called boundary lubrication.

5. When bearings must be operated at extreme temperature, asolid-film lubricant such as graphite or molybdenum disulfied must be used becouse the ordinary mineral oils are not statisfactory.

* Journal Bearing

The journal bearing consists of two rigid cylinders, the outer cylinder which is held Stationary is called bearing, while the inner cylinder which rotates at an angular velocity "w" is called the journal. The clearance space between bearing and shaft is filled with a lubricant, usually a petroleum oil.

W is carried by the oil pressure, generated by the rotational flow (due to viscosity) with the shaft, i.e., journal, rotation.

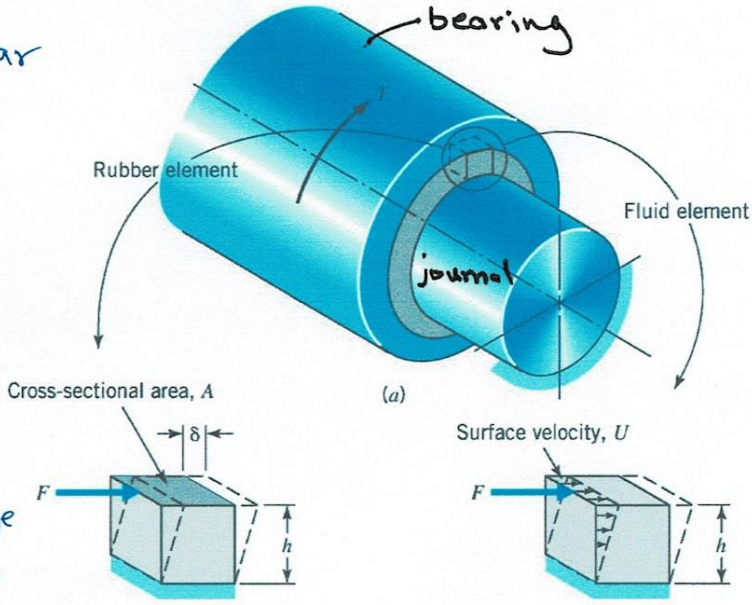


12.2 Viscosity

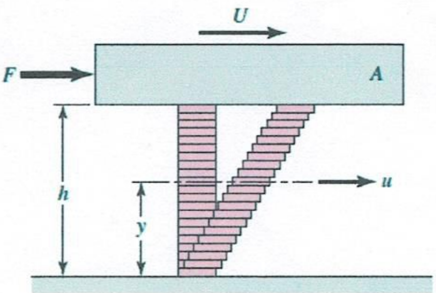
Think about shear stress

$$\tau = \frac{F}{A} = \mu \frac{du}{dy}$$

$$= \mu \frac{U}{h} \quad (1)$$



Where the derivative $\frac{du}{dy}$ is the rate of change of velocity with distance and may be called the rate of shear or the velocity gradient. For most lubricating fluids, the rate of shear is constant $\frac{du}{dy} = \frac{U}{h}$



$u=0$ at $y=0$

Rearrange into viscosity

$$\mu = \frac{\tau \cdot h}{U}$$ (absolute viscosity, dynamic viscosity)

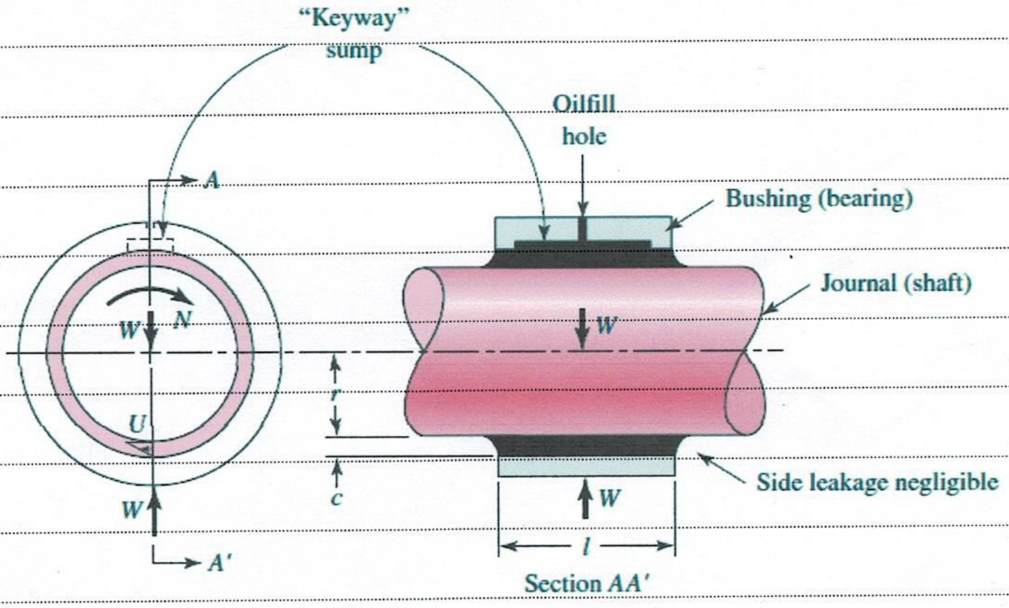
Kinematic viscosity $\upsilon = \frac{\mu}{\rho}$
where ρ is lubricant density.

Viscosity affected by temperature and pressure

$$\left.\begin{matrix} T \uparrow \\ \rho \downarrow \end{matrix}\right\} = \mu \downarrow$$

12.3 Petroff's Equation.

The phenomenon of bearing friction was first explained by Petroff on the assumption that the shaft is concentric. Let us now consider a vertical shaft rotating in a guide bearing. It is assumed that the bearing carries a very small load, that the clearance space is completely filled with oil, and that leakage is negligible, Figure below.



If the shaft rotates at N rev/s, then its surface velocity is $U = 2\pi r N$ (in/s). Since the shearing stress in the lubricant is equal to the velocity gradient times the viscosity,

$$\tau = \mu \frac{U}{h} = \frac{2\pi r \mu N}{C} \qquad (2)$$

Where the radial clearance "C" has been substituted for the distance "h".

The force required to shear the film is the stress time the area. The torque is the force times the lever arm 'r':

$$T = (\tau \cdot A)(r) = \left(\frac{2\pi r \mu N}{c}\right)(2\pi r L)(r) = \frac{4\pi^2 r^3 L \mu N}{c} \quad (3)$$

If we now designate a small force on the bearing by "W", in Newtons, then the pressure "P", in Pascal, is $P = \frac{W}{2rL}$.
The frictional force is "fW", where "f" is the coefficient of friction, and so the frictional torque is,

$$T = fWr = (f)(2rLP)(r) = 2r^2 f \cdot L \cdot P \quad (4)$$

Substituting the value of the torque from Eq. 3 in Eq. 4 and solving for the coefficient of friction, we find

$$f = 2\pi^2 \frac{\mu N r}{P c} \quad (5)$$

Equation (5) is called Pettroff's equation. The two quantites $\mu N/P$ and r/c are very important parameters in lubrication. Substitution of the appropriate dimensions in each parameter will show that they are dimensionless.

The bearing characteristic number, or the sommerfeld number, is defined by the equation

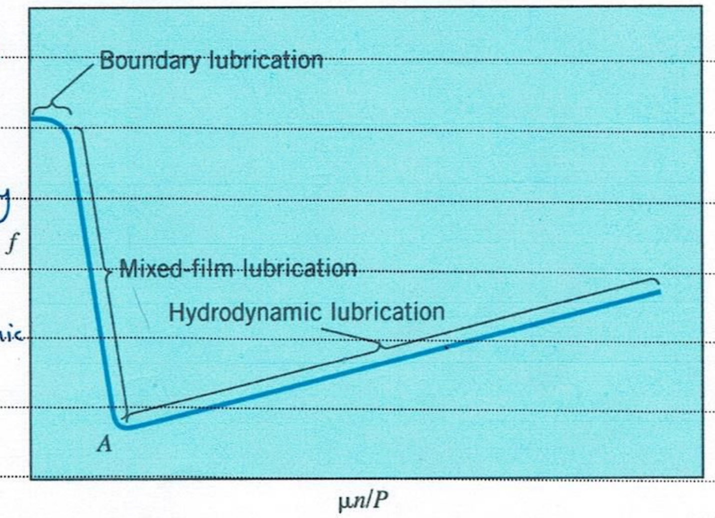
$$S = \left(\frac{r}{c}\right)^2 \cdot \frac{\mu N}{P} \quad (6)$$

The sommerfeld number is very important in lubrication analysis because it contains many of the parameters that are specified by the designer. Note that it is also dimensionless. The quantity "r/c" is called the "radial clearance ratio". If we multiply both sides of Eq. 5 by this ratio, we obtain the interesting relation

$$f\left(\frac{r}{c}\right) = 2\pi^2 \cdot \frac{\mu N}{P}\left(\frac{r}{c}\right)^2 = 2\pi^2 S \qquad (7)$$

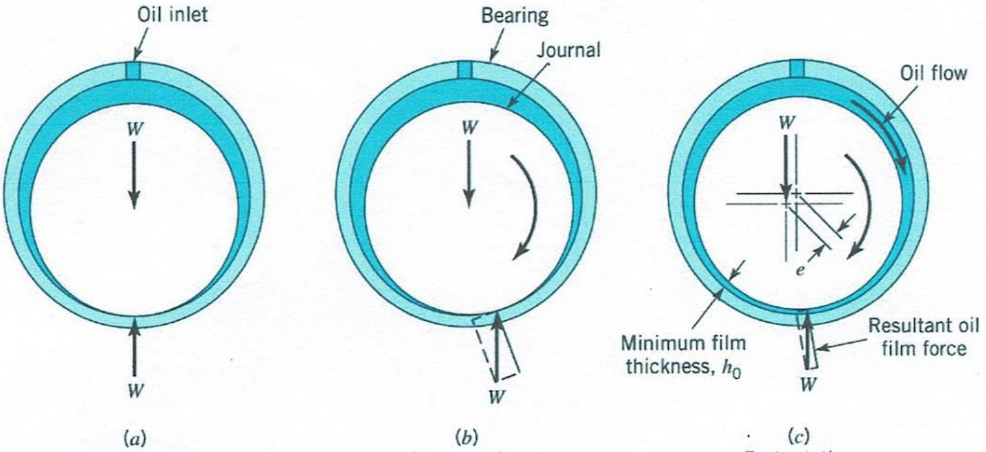
12.4 Stable Lubrication

The plot is important because it defines stability of lubrication and helps us to understand hydrodynamic and boundary, or thin-film, lubrication.



- friction lowest at A
- Thick-film lubrication, that is, no metal-to-metal contact, the surfaces being completely separated by a lubricant film.
- The region to the right of A defines stable lubrication because variation are self-correcting.
- Small viscosity, and hence a small $\mu N/P$, means that the lubrication film is very thin and that there will be a greater possibility of some metal-to-metal contact, and hence of more friction.

12.5 Thick-Film Lubrication

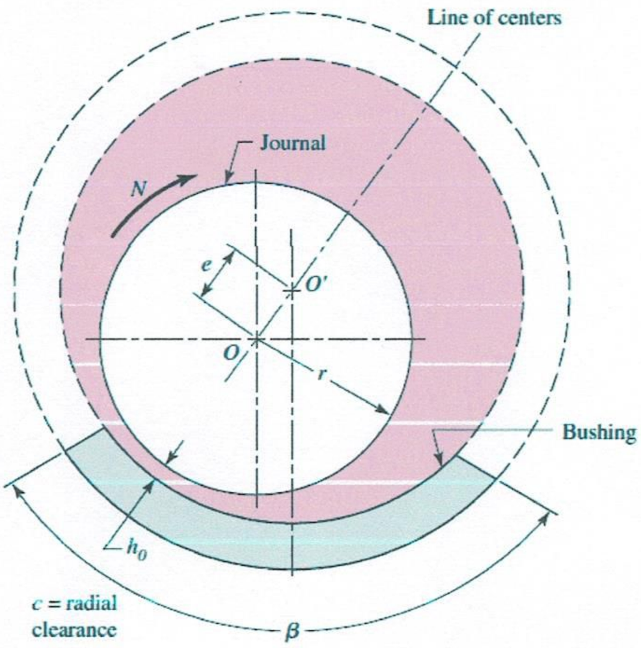


(a) At rest
(b) Slow rotation (boundary lubrication)
(c) Fast rotation (hydrodynamic lubrication)

To examine the formation of a lubricant film in a journal bearing, figure above shows a journal that is just beginning to rotate in a clockwise direction. Under starting conditions, the bearing will be dry, or at least partly dry, and hence the journal will climb or roll up the right side of the bearing as shown in Fig. b.

The action of the rotating journal is to pump the lubricant around the bearing in a clockwise direction. A minimum film thickness "h_o" occurs, not at the bottom of the journal, but displaced clockwise from the bottom as in figure (c). This is explained by the fact that a film pressure in the converging half of the film reaches a maximum somewhere to the left of the bearing center.

The nomenclature of a journal bearing is shown in figure.



c = radial clearance and is the difference in the radii of the bushing and journal ($r_{bearing\ or\ bush} - r_{journal\ or\ shaft}$).

O' = center of the bearing "fixed"

O = center of the journal "moving"

e = eccentricity " shortest distance between center of journal and center of bearing

h_o = minimum film thickness , $h_o = c - e$

N = rotational speed of the journal

r = journal radius

h = film thickness

$\epsilon = e/c$ dimensionless eccentricity ratio

$h_{max} = c + e$

The bearing shown in the figure above is known as a partial bearing. If the radius of the bushing is the same of the radius of the journal, it is known as a fitted bearing. If the busing encloses the journal, as indicated by the dashed lines, it becomes a full bearing. The angle β describes the angular length of a partial bearing.

12.6 Hydrodynamic Theory

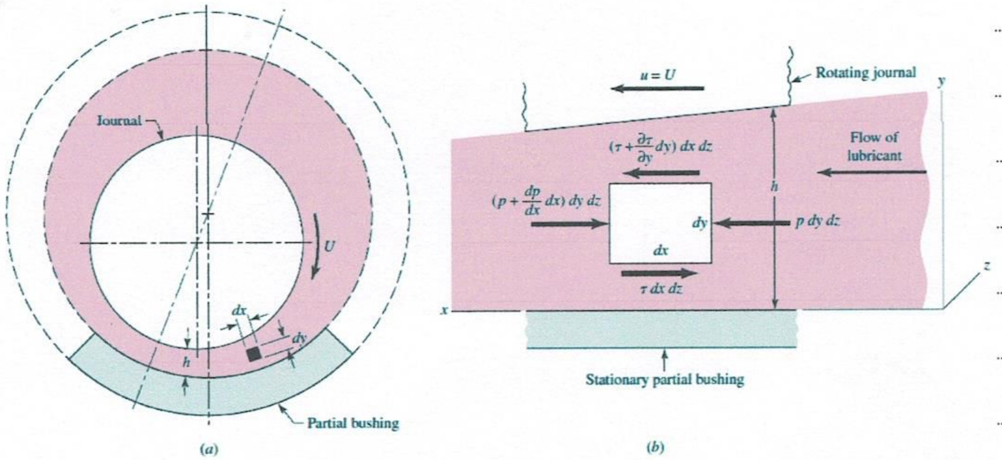
The present theory of hydrodynamic lubrication originated in the laboratory by Reynolds. Reynolds concluded that there must be a definite equation relating the friction, the pressure, and the velocity. Reynolds pictured the lubricant as adhering to both surfaces and being pulled by the moving surface into a narrowing, wedge-shaped space so as to create a fluid or film pressure of sufficient intensity to support the bearing load. One of the important simplifying assumptions resulted from Reynold's realization that the fluid films were so thin in comparison with the bearing radius that the curvature could be neglected. This enabled him to replace the curved partial bearing with a flat bearing, called a plane slider bearing. Other assumptions made were:

1. The lubricant obeys Newton's viscous effect.
2. The forces due to the inertia of the lubricant are neglected.
3. The lubricant is assumed to be incompressible.
4. The viscosity is assumed to be constant throughout the film.
5. The pressure does not vary in the axial direction.

Figure below shows a journal rotating in the clockwise direction supported by a film of lubricant of variable thickness "h" on a partial bearing, which is fixed. We specify that the journal has a constant surface velocity "U". Using Reynold's assumption that curvature can be neglected. Additional assumptions can be made.

6. The bushing and journal extend infinitely in the Z-direction; this means there can be no lubricant flow in the Z-direction.

7. The film pressure is constant in the y-direction. Thus the pressure depends only on the coordinate x.
8. The velocity of any particle of lubricant in the film depends only on the coordinate x and y.



Now, by slecting an element of lubricant in the film Fig. a above of diamensions dx, dy, and dz, and compute the forces that act on the sides of this element. As shown in Fig. b, normal forces, due to the pressure, act upon the right and left sides of the element, and shear forces, due to the viscosity and to the velocity, act upon the top and bottom sides. Summing the forces in the x-direction gives;

$$\sum F_x = p\,dy\,dz - \left(p + \frac{dP}{dx}dx\right)dy\,dz - \tau\,dx\,dz + \left(\tau + \frac{\partial \tau}{\partial y}dy\right)dx\,dz = 0 \quad (8)$$

This reduce to

$$\frac{dP}{dx} = \frac{\partial \tau}{\partial y}$$

Recall equ $\tau = \mu \frac{\partial u}{\partial y}$

$$\Rightarrow \frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}$$

Holding x constant, by integrating this expression twice with respect to y.

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{dp}{dx} y + C_1$$

$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + C_1 y + C_2$$

By assuming there is no slip between the lubricant and the boundary surfaces. At $y=0 \Rightarrow u=0$ & At $y=h \Rightarrow u=U$

$$\Rightarrow C_2 = 0 \quad \& \quad C_1 = \frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx}$$

$$\therefore u = \frac{1}{2\mu}\frac{dp}{dx} \cdot y^2 + \left(\frac{U}{h} - \frac{h}{2\mu}\cdot\frac{dp}{dx}\right)\cdot y$$

$$= \frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh) + \frac{U}{h} y$$

When the pressure is maximum $\Rightarrow \frac{dp}{dx} = 0$

$$\Rightarrow u = \frac{U}{h} y$$

To find the volume of lubricant flowing in the x-direction per unit time. By using a width of unity in the Z-direction, the volume may be obtained by the expression

$$Q = \int_0^h u \cdot dy$$

Substituting the value of u and integrating

$$\Rightarrow Q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dP}{dx}$$

For incompressible lubricant and states that the flow is the same for any cross section. Thus

$$\frac{dQ}{dx} = 0$$

$$= \frac{U}{2}\frac{dh}{dx} - \frac{d}{dx}\left(\frac{h^3}{12\mu} \cdot \frac{dP}{dx}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{h^3}{\mu}\frac{dP}{dx}\right) = 6U \cdot \frac{dh}{dx}$$

Which is the classical Reynolds equation for one-dimensional flow. It neglects side leakage, that is, flow in the Z-direction. A similar development is used when side leakage is not neglected. The resulting equation is;

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial P}{\partial z}\right) = 6U \cdot \frac{\partial h}{\partial x}$$

There is no general analytical solution to above equation; approximate solution have been obtained. One of the important solutions is due to Sommerfeld and may be expressed in the form

$$\frac{r}{c} f = \phi\left[\left(\frac{r}{c}\right)^2 \frac{\mu N}{P}\right]$$

where ϕ indicates a functional relationship. Sommerfeld found the function for half and full bearings by assuming no side leakage.

12.7 Design Considerations

We may distinguish between two groups of variables in the design of sliding bearings. In the first group are those whose values either are given or are under control of the designer. These are:

1. The viscosity, μ
2. The load per unit of projected bearing area, P
3. The speed, N
4. The bearing dimensions, r, c, β, and l

In the second group are the dependent variables. The designer cannot control these except indirectly by changing one or more of the first group. These are:

1. The coefficient of friction, f
2. The temperature rise, ΔT
3. The volume flow rate of oil, Q
4. The minimum film thickness, h_o

To this point we have assumed that only the journal bearing rotates and it is the journal rotational speed that is used in the Summerfeld number. It has been discovered that the angular speed "N" that is significant to hydrodynamic film bearing performance is

$$N = |N_j + N_b - 2N_f|$$

Where: N_j = journal angular speed, rev/s

N_b = bearing angular speed, rev/s

N_f = load vector angular speed, rev/s

Figure 12-11 pg 610 shows several situations for determining N.

12.8 The Relations of the Variables

Before proceeding to the problem of design, it is necessary to establish the relationships between the variables.

- **Viscousity Charts**

Figures 12-12 to 12-14 will be used to determine the viscosity corresponding to any given temperature in analysing bearings.

$$T_{av} = T_1 + \frac{\Delta T}{2}$$

Where: T_1 = the inlet temperature

ΔT = the temperature rise of the lubricant from inlet to outlet $(T_{out} - T_{in})$.

Of course, the viscosity used in the analysis must correspond to T_{av}. One of the objective of lubrication analysis is to determine the oil outlet temperature when the oil and its inlet temperature are specified. This is a trial-and-error type of problem. In an analysis temperature rise will first be estimated. This allows for the viscosity to be determine from the chart. With the value of the viscosity, the analysis is performed where the temperature rise is then computed. With this, a new estimate of the temperature rise is established. This process is continued until the estimated and computed temperatures agree.

• Minimum Film Thickness and Eccentricity Ratio

In Figure 12-16, the minimum film-thickness variable h_o/c and eccentricity ratio $\epsilon = e/c$ are plotted against the Sommerfeld "S" with contours for various values of l/d.

- The minimum film thickness variable $\left(\frac{h_o}{c}\right)$ which is dimensionless
- The eccentricity ratio $\epsilon = \frac{e}{c}$ which is dimensionless. Where "e" is the eccentricity which is the distance between the center of bearing and the center of journal.

The corresponding angular position of the minimum film thickness is found in Fig. 12-17.

Design optima are sometimes maximum load, which is a load-carrying characteristic of bearing, and sometimes minimum parastic power loss or minimum coefficient of friction. Dashed lines appear on figure 12-16 for maximum load or minimum coefficient of friction, so you can easily favor one of maximum load or minimum coefficient of friction, but not both. The Zone between the two dashed-line contours might be considered a desirable location for a design point.

• Coefficient of Friction "f"

The friction chart, Fig. 12-18 has the friction variable $(r/c)f$ plotted against Sommerfeld number S with contours for various values of the l/d ratio.

$$f = \frac{F}{W} = \frac{\text{frictional force}}{\text{load}}$$

∴ Frictional torque $T = F \cdot r$

So, the power lost in the bearing is,

$$H = T \cdot \omega \qquad (\omega = \text{rad/s})$$

• Lubricant Flow

The flow variable $(Q/r.c.N.L)$ found from the Fig. 12-19 which is used to find the volume of lubricant (Q) which is pumped into the converging space by the rotating journal. Where (Q) is the lubricant flow rate inside the bearing.

If the amount of oil (Q) pumped by the rotating journal, an amount (Q_s) flows out the ends and is called the side leakage. This value can be computed from the flow ratio (Q_s/Q) from the Fig. 12-20.

• Film Pressure

The maximum pressure developed in the film can be estimated by finding the pressure ratio P/P_{max} from the chart in Fig. 12-21. The locations where the terminating and maximum pressures occur, as defined in Fig. 12-15, are determined from Fig. 12-22.

• Lubricant Temperature Rise

The journal does work on the lubricant which produces a heat. This heat must be dissipated out and carried away by the flow of oil. If we assume that the temperature of the side flow is the mean of the inlet and outlet temperatures, the temperature rise of the side flow (Q_s) is $\Delta T_c/2$. Then the heat generated raises the temperature of the flow ($Q-Q_s$) an amount (ΔT_c). The following equation determine the temperature rise;

$$\frac{0.12\,\Delta T_c}{P_{Mpa}} = \frac{rf/c}{\left(1-\frac{1}{2}Q_s/Q\right)\left[Q/rcN_j l\right]}$$

Where: Q = volumetric oil-flow rate into the bearing, m^3/s

Q_s = Volumetric side-flow leakage rate out of the bearing and to the sump, m^3/s

ΔT_c = the temperature rise in oil between inlet and outlet, $^\circ C$

T_1 = oil inlet temperature ; $^\circ C$

$Q-Q_s$ = volumetric oil-flow discharge from annulus to sump, m^3/s

P_{Mpa} = the bearing pressure in Mpa.

Example:

A full journal bearing has a nominal diameter of 50mm and a bearing length of 25 mm. The bearing supports a load of 3000N and the journal design speed is 3000 rpm. The radial clearence has been specified as 0.04 mm An SAE 10 oil has been chosen and the lubricant supply temperature is 50°C. Find the temperature rise of the lubricant, the lubricant flow rate, the minimum film thickness, the torque required to overcome friction and the heat generated in the bearing.

Solution:

let $\Delta T = 20°C$

$$T_{av} = T_1 + \frac{\Delta T}{2} = 50 + \frac{20}{2} = 60°C$$

From Fig. 12-13 for SAE 10 at 60°C $\Rightarrow \mu = 0.013$ pa.s

$N = \frac{3000}{60} = 50$ rps , $L/D = \frac{1}{2}$

$$P = \frac{W}{LD} = \frac{3000}{0.025 * 0.05} = 2.4 * 10^6 \ N/m^2 = 2.4 \ Mpa$$

$$\Rightarrow S = \left(\frac{r}{c}\right)^2 \frac{\mu N}{P} = \left(\frac{25*10^{-3}}{0.04*10^{-3}}\right)^2 \frac{0.013*50}{2.4*10^6} = 0.105$$

For $S = 0.105$ and $\frac{L}{D} = \frac{1}{2}$

$\Rightarrow$ From Fig. 12-18 $\Rightarrow \frac{r}{c} f = 3.65$

$\Rightarrow$ From Fig 12-19 $\Rightarrow Q/(rcNL) = 5.45$

$\Rightarrow$ From Fig 12-20 $\Rightarrow Q_s/Q = 0.875$

$$\Delta T = \frac{P_{MPa}}{0.12(1 - 0.5(Q_s/Q))} \cdot \frac{(r/c) f}{[Q/(rCNL)]}$$

$$= \frac{2.4}{0.12[1 - 0.5(0.87)]} \cdot \frac{3.6}{5.4} = 23.7\ ^{\circ}C$$

$$T_{av} = T_1 + \frac{\Delta T}{2} = 50 + \frac{23.7}{2} = 61.85\ ^{\circ}C$$

Repeating the procedure Using the new value for Tav gives $\mu = 0.0125$ pa.s from Fig 12-13.

$\Rightarrow S = \left(\frac{r}{c}\right)^2 \frac{\mu N}{P} = 0.1$

$\Rightarrow \therefore$ From Fig 12-18 $\Rightarrow \frac{r}{c} f = 3.6$

From Fig 12-19 $\Rightarrow Q/(rCNL) = 5.4$

From fig 12-20 $\Rightarrow Qs/Q = 0.87$

$\Delta T = 23.5$

$$T_{av} = T_1 + \frac{\Delta T}{2} = 50 + \frac{23.5}{2} = 61.75\ ^{\circ}C$$

This value for Tav is close to the previous Calculated value. $\therefore T_{av} = 61.75\ ^{\circ}C$; $\mu = 0.0125$ pa.s and $S = 0.1$.

The other parameters Can now be found.

$Q = rCNL * 5.4 = 25 * 0.04 * 50 * 25 * 5.4$

$= 6750.5\ mm^3/s$

$\Rightarrow$ From Fig. 12-16 $\Rightarrow \frac{h_o}{c} = 0.21 \Rightarrow h_o = 0.21 * 0.04 = 0.0084$ mm

$f = 3.6 * \frac{c}{r} = 3.6 * \frac{0.04}{25} = 0.00576$.

The torque is given by.

$Torq = f \cdot W \cdot r = 0.432$

$H = 2\pi N * Torque = 2\pi * \frac{3000}{60} * 0.432 = 135.7$ watt.

12.9 Steady-State Conditions in Self-Contained Bearings

The case in which the lubricant carries away all of the enthalpy increase from the journal-bushing pair has already been discussed. Bearing in which the warm lubricant stays within the bearing housing will now be addressed. These bearings are called self-contained bearings because the lubricant sump is within the bearing housing and the lubricant is cooled within the housing. These bearings are described as pillow-block or pedestal bearings. The heat given up by the bearing housing may be estimated from the equation.

$$H_{loss} = h_{CR} A (T_b - T_\infty) \qquad (1)$$

Where H_{loss} = heat dissipated, J/s or W

h_{CR} = Combined overall coefficient of radiation and convection heat transfer, $W/(m^2 \cdot {}^\circ C)$

A = surface area of bearing housing, m^2

T_b = surface temperature of the housing, °C

T_∞ = ambient temperature, °C

The overall coefficient "h_{CR}" depends on the material, surface coating, geometry, even the roughness, the temperature difference between the housing and surrounding objects, and air velocity. The overall coefficient can be treated as a constant.

$$h_{CR} = \begin{cases} 11.4 \ W/(m^2 \cdot {}^\circ C) & \text{for still air} \\ 15.3 \ W/(m^2 \cdot {}^\circ C) & \text{for shaft-stirred air} \\ 33.5 \ W/(m^2 \cdot {}^\circ C) & \text{for air moving at } 25.4 \ m/s \end{cases}$$

Eq. 1 can be written for the temperature difference $T_f - T_b$ between the lubricant film and the housing surface. This is possible because the bushing and housing are metal and very nearly isothermal. The following proportionality has been observed

$$\bar{T}_f - T_b = \alpha (T_b - T_\infty) \quad (2)$$

where $\bar{T}_f$ = average film temperature ($\bar{T}_f = T_s + \Delta T/2$)

T_s = lubricant inlet temperature

α = constant depending on the lubrication scheme and the bearing housing geometry, Table (12-2).

solving Eq. 2 for T_b and subsituting into Eq. 1 gives,

$$H_{loss} = \frac{h_{CR} \cdot A}{1+\alpha} (\bar{T}_f - T_\infty)$$

In the beginning a steady-state analysis the average film temperature is unknown, hence the viscosity of the lubricant in a self-contained bearing is unknown. Finding the equilibrium temperatures is an iterative process wherein a trial average film temperature (and the corresponding viscosity) is used to compare the heat generation rate and the heat loss rate. An adjustment is made to bring these two heat rates into agreement. The heat generation rate H_{gen}, at steady state, is equal to the work rate from the frictional torque (T).

$$H_{gen} = 2\pi T N \quad (3)$$

where $T = 4\pi^2 r^3 l \mu N / c$ (4)

$$\therefore H_{gen} = \frac{4\pi^2 r^3 l \mu N}{c}(2\pi N) = \frac{248 \cdot \mu \cdot N^2 \cdot l \cdot r^3}{c}$$

$H_{loss} = H_{gen}$

$$\Rightarrow \frac{248 \cdot \mu \cdot N^2 \cdot l \cdot r^3}{c} = \frac{h_{CR} \cdot A}{1+\alpha}(\bar{T}_f - T_\infty)$$

$$\therefore \bar{T}_f = T_\infty + 248(1+\alpha)\frac{\mu \cdot N^2 \cdot l \cdot r^3}{h_{CR} \cdot A \cdot c}$$

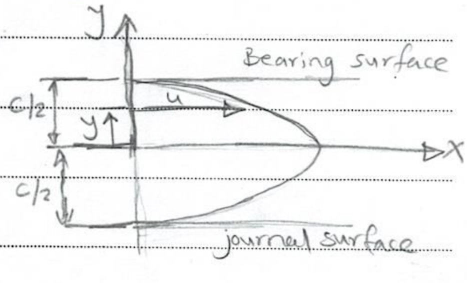
12.10 Pressure-Fed Bearings

The load-carrying capacity of self-contained natural-circulating journal bearing is quite restricted. A first through of a way to increase heat dissipation is ~~to~~ cool the sump with an external fluid such as water. A second alternative is to reduce the temperature rise in the film by dramatically increasing the rate of lubricant flow. To increase lubricant flow, an external pump must be used. Because the lubricant is supplied to the bearing under pressure, such bearing are called pressure fed bearing.

Figure 12-28 shows the flow of the lubricant is caused by the supply pressure p_s. Laminar flow is assumed, with the pressure varying linearly from $p = p_s$ at $x = 0$, to $p = 0$ at $x = l'$.

$$u = \frac{p_s}{8\mu l'}(c^2 - 4y^2)$$

at $y = 0 \Rightarrow u_{max} = \frac{p_s \cdot c^2}{8\mu l'}$



The amount of lubricant that flow out both ends:

$$Q_s = \frac{\pi \cdot P_s \cdot r \cdot c^3}{3\mu l'} (1 + 1.5\,\epsilon^2)$$

The characteristic pressure in each of the two bearings is

$$P = \frac{W/2}{2rl'} = \frac{W}{4rl'}$$

The Sommerfeld number

$$S = \left(\frac{r}{c}\right)^2 \frac{\mu N}{P} = \left(\frac{r}{c}\right)^2 \frac{4rl'\mu N}{W}$$

$$\Delta T_c = \frac{978\,(10^6)}{1 + 1.5\,\epsilon^2} \, \frac{(fr/c) \cdot S \cdot W^2}{P_s\, r^4}$$

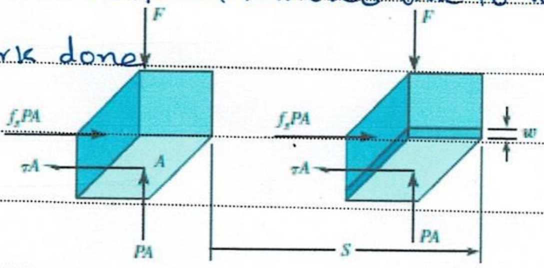
$$H_{gain} = \rho \cdot C_p \cdot Q_s \cdot \Delta T$$

12-11 Thrust Bearings

A thrust bearing is a particular type of rotary bearing. Like other bearings they permit rotation between parts, but they are designed to support a predominantly axial load. Table 12-7 gives some properties of a range of bushing materials.

• Linear Sliding Wear

Consider the sliding block depicted in Fig. below, moving along a plate with contact pressure "p" acting over area "A", in the presence of a cofficient of sliding friction f_s. The material removed due to wear is wA and is proportional to the work done



$$wA \propto f_s P A V t$$

$$wA = K P A V t$$

$$w = K P V t$$

Where w = linear measur of wear, mm

K = proportional factor, includes f_s, and is determined from laboratory testing, Tables 12-8 and 12-9

t = time

It is useful to include a modifying factor f_1 depending on motion type, load, and speed and an environment factor f_2 to account for temperature and cleanliness conditions, (Tables 12-10 and 12-11). These factors account for departures from the laboratory conditions under which K was measured

$$\therefore \quad w = f_1 f_2 K \cdot p \cdot V \cdot t$$

• Bushing Wear

Consider a pin of diameter D, rotating at speed N, in a bushing of length L, and supporting a stationary radial load F. The nominal pressure P is given by

$$P = \frac{F}{DL}$$

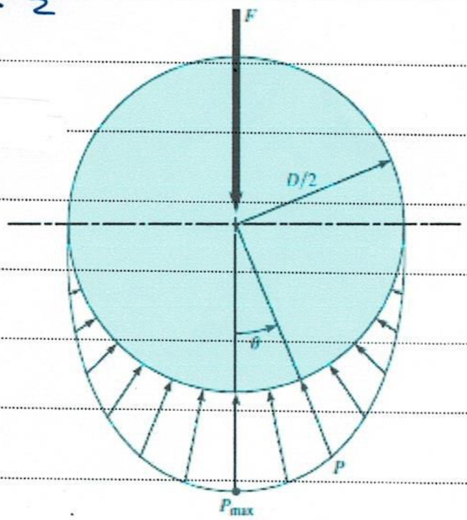
Thus,

$$PV = \frac{F}{DL}\,\pi D N = \pi\,\frac{FN}{L}$$

where V = velocity in m/s , $V = 2\pi r N = \pi D N$

Figure below provides a more accurate representation of the pressure distribution, which can be written as

$$P = P_{max}\cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$



The vertical component of p dA is

$$p\,dA\cos\theta = [PL(D/2)\,d\theta]\cos\theta$$

$$= P_{max}(DL/2)\cos^2\theta\,d\theta$$

Integrating this form

$$\int_{-\pi/2}^{\pi/2} P_{max}\left(\frac{DL}{2}\right)\cos^2\theta\,d\theta = \frac{\pi}{4}P_{max}DL = F$$

$$\therefore P_{max} = \frac{4}{\pi}\frac{F}{DL}$$

But $\omega = f_1 f_2 K P V t$

$$\therefore \omega = f_1 \cdot f_2 \cdot K \cdot \frac{4}{\pi}\cdot\frac{F}{DL}\cdot\pi D N t = 4\,\frac{f_1 f_2 K F N t}{L}$$

In designing a bushing, the recommended length/diameter ratio is

$$0.5 \le L/D \le 2$$

• Temperature Rise

At steady state, the rate at which work is done against bearing friction equals the rate at which heat is transferred from the bearing housing to the surroundings by convection and radiation. The rate of heat generation in J/s.

$$H_{gen} = \frac{f_s\, F V}{J} = \frac{f_s\, F\,(\pi D)(60N)}{J} = \frac{\pi f_s F D N}{J}$$

Where N = journal speed, rev/s

$J = 1\ m.N/J$

$$H_{loss} = \hbar_{CR}\, A\, \Delta T = \hbar_{CR}\, A\,(T_b - T_\infty) = \frac{h_{CR} A}{2}(T_f - T_\infty)$$

Where A = housing surface area, m^2

$\hbar_{CR}$ = overall combined coefficient of heat transfer, $W/(m^2 \cdot {}^\circ C)$

T_b = housing metal temperature, °C

T_f = lubricant temperature, °C

T_∞ = ambient temperature, °C

Equating the two equations above

$$T_f = T_\infty + \frac{2\pi f_s F D N}{J\, \hbar_{CR}\, A}$$

If the bushing is to be housed in pillow block, the surface area can be roughly estimated from

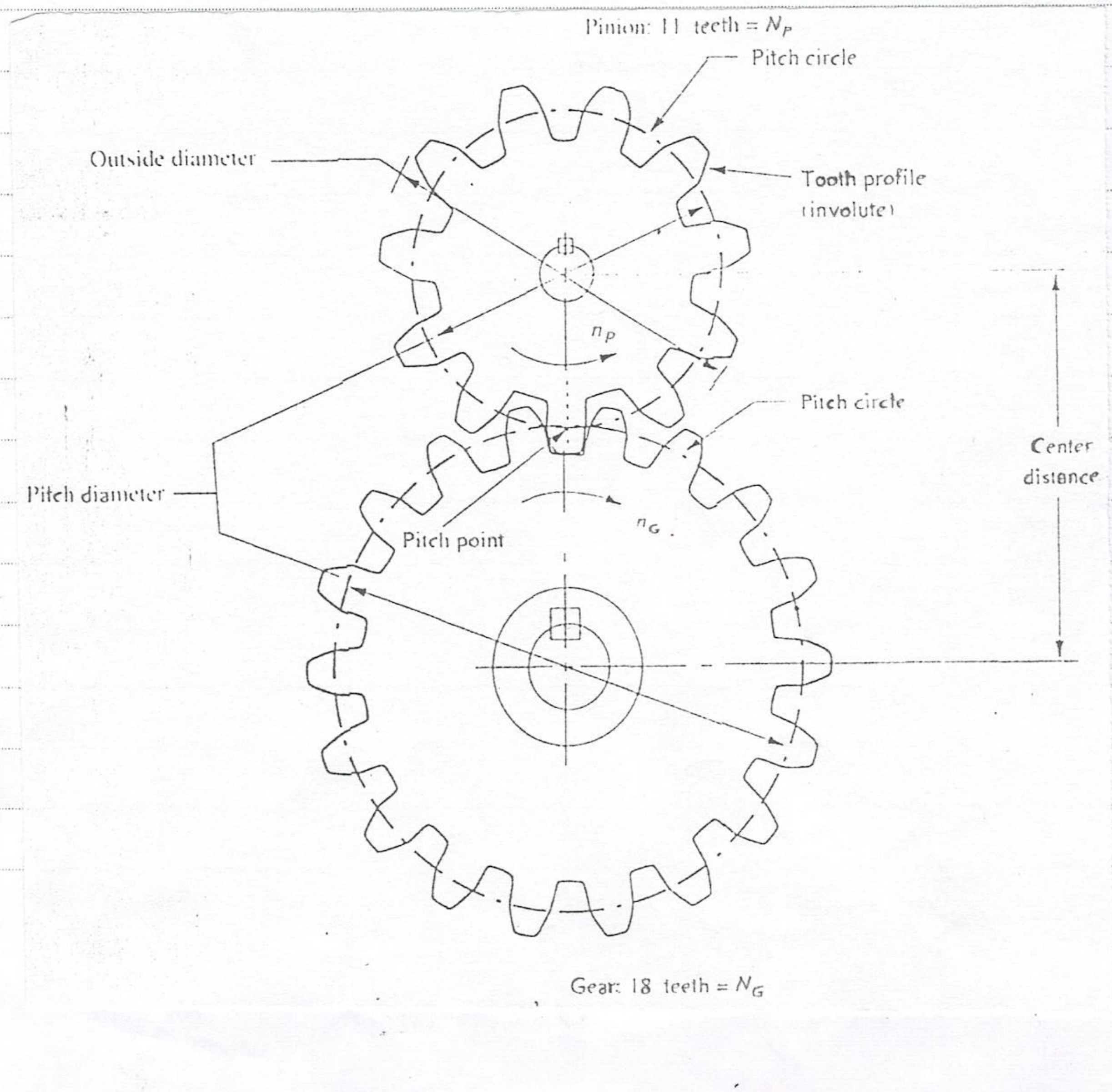
$$A = 2\pi . D . L$$

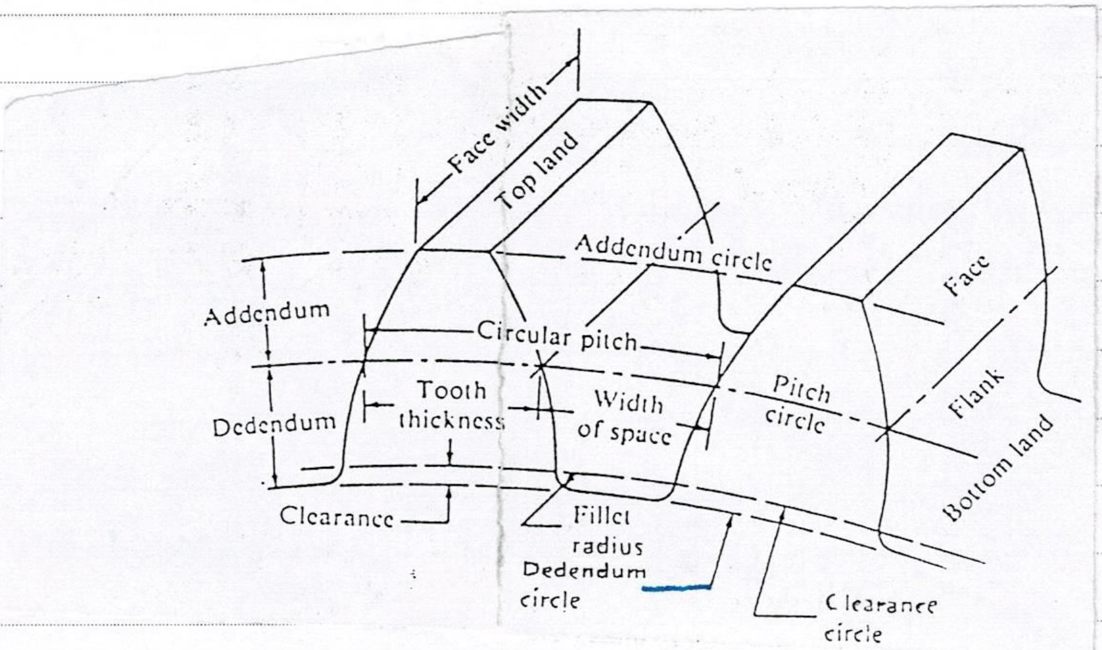
$$\therefore\ T_f = T_\infty + \frac{f_s F N}{J\, \hbar_{CR}\, L}$$

CHAPTER 13
Spur Gears

1. Nomenclature.

• spur gear have teeth that are straight and arranged parallel to the axis of the shaft, that carries the gear. The curved shape of the faces of the spur gear teeth have a special geometry called an involute curve. The spur gears used to transmit rotary motion and power between parallel shafts.





• The pitch circle is a theoretical circle upon which all calculations are usually based. The pitch circles of a pair of mating gears are tangent to each other. A pinion is the smaller of two mating gears. The larger is often called the gear.

• The circular pitch (P_c) is the distance between two symmetric points lie on the two alternatively teeth, so

$$P_c = \frac{\pi d}{N}$$

where: d = pitch diameter mm.
N = number of teeth.

• The diametral pitch (P_d) is the ratio of the number of teeth on the gear to the pitch diameter, it is expressed as tooth per inch.

$$P_d = \frac{N}{d}$$

- The module (m) is the ratio of the pitch diameter to the number of teeth, so

$$m = \frac{d}{N}$$

- The addendum (a) is the radial distance between the top land and the pitch circle.

- The dedendum (b) is the radial distance between the bottom land and the pitch circle.

- The whole depth (ht) is the sum of the addendum and dedendum.

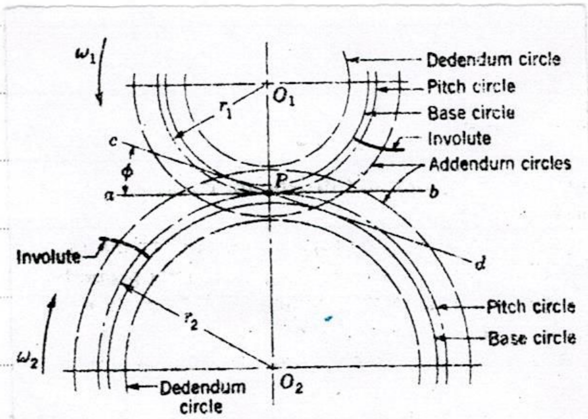
- The clearance circle is a circle that is tangent to the addendum circle of the mating gear.

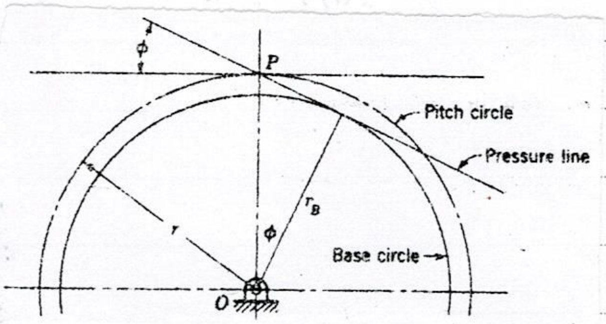
- The clearance (C) is the amount by which the dedendum in a given gear exceeds the addendum of its mating gear.

- The back lash is the amount by which the width of a teeth space axceeds the thickness of the engaging tooth measured on the pitch circles.

• pitch point is the point at which the pitch circles of the pinion and gear are tangent.

• the pressure angle is that angle generated between the tangent for the pitch circles of the two mating gears through the pitch point and the pressure line. It usually has values 20° and 25°.





• the base circle is that circle tangent to the pressure line. So,

$$r_b = r \cdot \cos\phi$$

where: r_B = radius of base circle.
r = pitch radius.
ϕ = pressure angle.

also;

$$P_B = P_C \cdot \cos\phi$$

where: P_B = Base circular pitch.
P_C = Circular pitch.

- The addendum and dedendum distances for standard respectively interchangeable teeth are m and 1.25 m . And to draw a tooth we must know the tooth thickness, so;

$\because P_c = \pi . m$ then

$$t = \frac{P_c}{2}$$

where:

P_c: circular pitch.

t : tooth thickness.

When two gears are in mesh, their pitch circles roll on one another with out slipping. If the pitch radii as r_1 and r_2 and the angular velocities as ω_1 and ω_2 respectively.

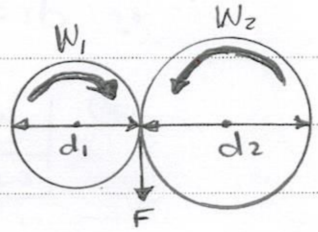
→ velocity analysis,

$\because V = \omega \cdot r$

$\Rightarrow V_1 = \omega_1 \cdot r_1$

$V_2 = \omega_2 \cdot r_2$

But $V_1 = V_2$



$$\Rightarrow \frac{W_1}{W_2} = \frac{r_2}{r_1}$$

$$\Rightarrow \frac{W_1}{W_2} = \frac{d_2}{d_1}$$

→ force analysis.

$Power)_{Gear\ 1} = power)_{Gear\ 2}$

$T_1 \cdot \omega_1 = T_2 \cdot \omega_2$

$$T_1 \cdot \frac{2\pi W_1}{60} = T_2 \cdot \frac{2\pi W_2}{60}$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{W_2}{W_1} = \frac{d_1}{d_2}$$

or, $T_1 = F * \frac{d_1}{2}$, $T_2 = F * \frac{d_2}{2}$

$$\Rightarrow \frac{T_1}{T_2} = \frac{F * d_1/2}{F * d_2/2} \Rightarrow \frac{T_1}{T_2} = \frac{d_1}{d_2}$$

Example: A 19-tooth pinion has a module of 2.5mm and runs at a speed of 1740 rpm. The driven gear is to operate at about 470 rpm. Find the circular pitch, the number of teeth on the gear, and the theoretical center distance.

Solution.

$$\because \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} \Rightarrow \frac{1740}{470} = \frac{N_2}{19}$$

$$\Rightarrow N_2 = 70$$

$$m = \frac{d}{N} \Rightarrow 2.5 = \frac{d_1}{19} \Rightarrow d_1 = 47.5 \text{ mm}.$$

and $$2.5 = \frac{d_2}{70} \Rightarrow d_2 = 175 \text{ mm}.$$

$$\therefore L_c = \frac{d_1 + d_2}{2}$$

$$= \frac{47.5 + 175}{2} = 111.25 \text{ mm}.$$

$$P_c = \pi \cdot m$$
$$= \pi \cdot (2.5)$$
$$= 7.85 \text{ mm}.$$

Ans

2. Gear Trains.

Consider a pinion 2 driving a gear 3. The angular velocity of the gear is,

$$\omega_3 = \frac{N_2}{N_3} \cdot \omega_2$$

OR

$$\omega_3 = \frac{d_2}{d_3} \cdot \omega_2$$

where: ω = rpm or number of turns.
N = number of teeth.
d = pitch diameter.

for a gear train as shown, the speed of the last gear is;

$$\omega_6 = \frac{N_2}{N_3} \cdot \frac{N_3}{N_4} \cdot \frac{N_5}{N_6} \cdot \omega_2$$

idler gear

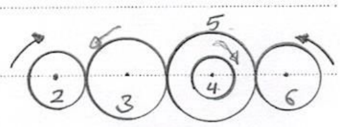


Figure 13 - 27

we define "train value (e)" as;

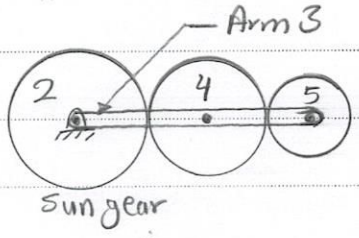
$$e = \frac{\text{product of driving tooth number}}{\text{product of driven tooth number}}$$

the train value (e) is postive if the last gear rotates in the same sense as the first, and negative if the last rotates in the opposite sense. Now can write.

$$\omega_L = e \ \omega_F$$

where: ω_L = the speed of the last gear.
ω_F = the speed of the first gear.

Unusual effects can be obtained in a gear train by permitting some of the gear axes to rotate about others. Such trains are called (planetary). It always include a sun gear, a planet carrier or arm, and one or more planet gears. For such trains;



$$e = \frac{W_L - W_A}{W_F - W_A}$$

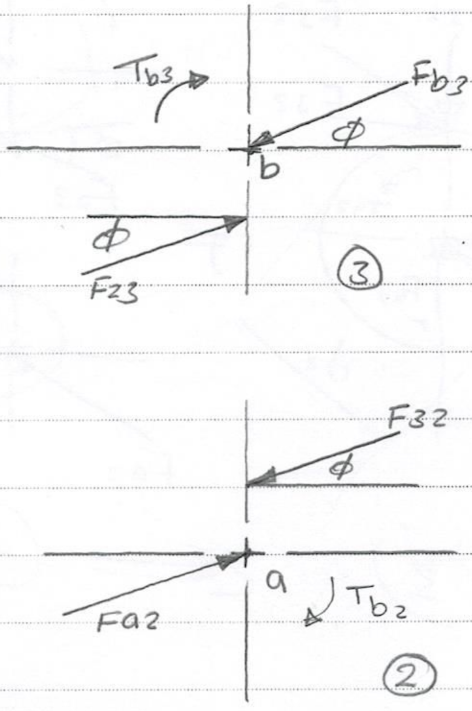
where:

W_F = rpm of first gear in planetary train.

W_L = rpm of last gear in planetary train.

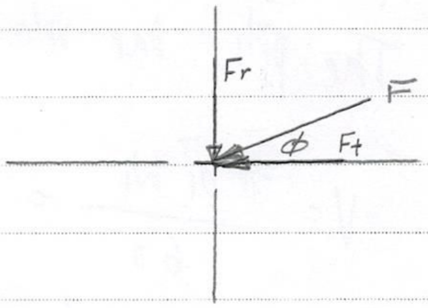
W_A = rpm of arm.

3. Force Analysis.



Here F_{23} is the force exerted by gear (2) aginst gear (3). The force of gear (2) against shaft (a) is F_{2a}. we can also write F_{a2} to mean the force of shaft (a) against gear (2).

$F_t \equiv$ Tangential Component.
$F_r \equiv$ Radial Component.



The transmitted load or useful load is the tangential Component, i.e:

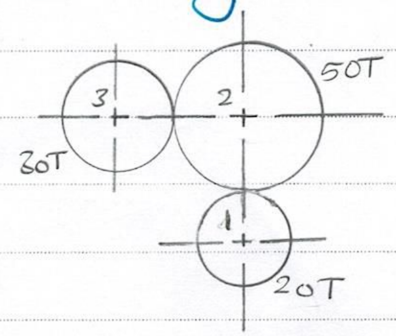
$$W_t = F_t$$

where: $W_t \equiv$ transmitted load.

- Applied torque $(T) = W_t \cdot \frac{d}{2}$, where: d = pitch dia.
- The pitch line velocity $(V) = \frac{\pi \cdot d \cdot W}{60}$
  where: V = m/s and w = speed rpm.
- The power transmitted $(H) = W_t \cdot V$

Ex 13-7

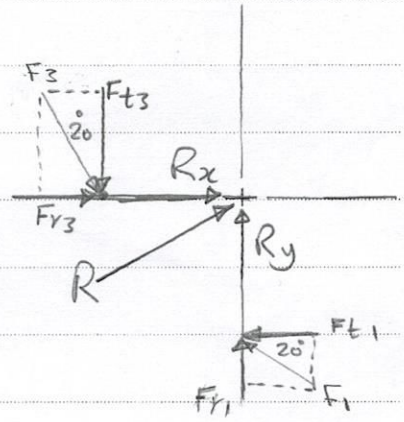
Example: A pinion runs at 1750 rev/min and transmits 2.5 kW to another pinion through an idler gear as shown. The teeth are cut on the 20°, Full depth system and have a module of $m = 2.5$ mm. Determine the reaction on the shaft of idler.



Solutions.

$d_1 = N_1 \cdot m = 20\,(2.5) = 50$ mm.

$d_2 = N_2 \cdot m = 50\,(2.5) = 125$ mm.



$$V = \frac{\pi \cdot d \cdot W}{60} = \frac{\pi \cdot (0.05) \cdot (1750)}{60}$$

$$= 4.58 \text{ m/s}$$

$$W_t = \frac{H}{V} = \frac{2500}{4.58} = 546 \text{ N.}$$

$\therefore F_{t_1} = F_{t_3} = 546$ N.

$F_{r_1} = F_{r_3} = 546 * \tan 20 = 199$ N.

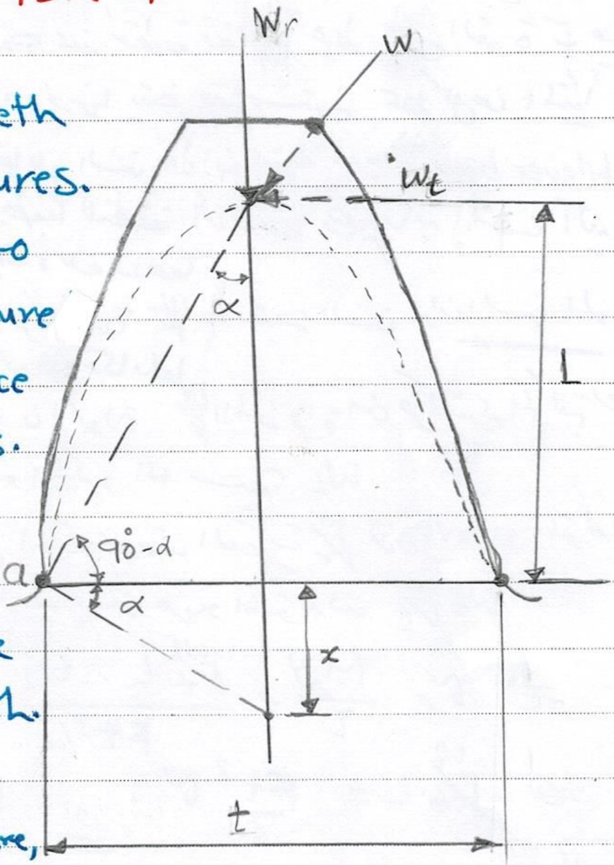
$\Rightarrow F_1 = F_3 = \dfrac{546}{\cos 20} = 581$ N.

$$R_x = 546 - 199 = 347 \text{ N} = R_y$$

$$\therefore R = \sqrt{R_x^2 + R_y^2} = 491 \text{ N.}$$

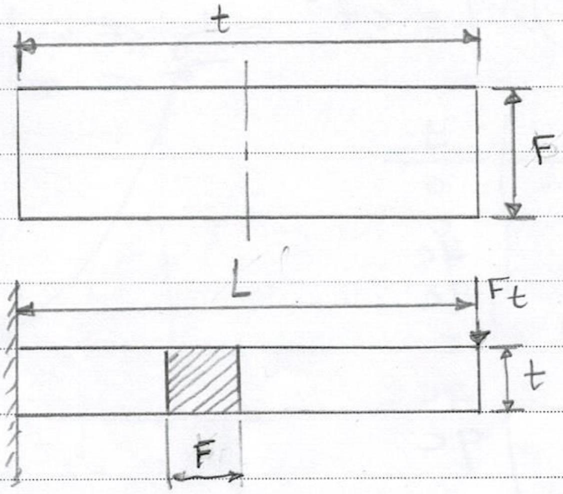
CHAPTER 14

1. Tooth Stresses.



The strength of gear teeth based on three kinds of failures. These are static failure due to bending stress, fatigue failure due to bending stress and surface fatigue due to contact stress.

Wilfred Lewis pressented a formula for computing the bending stress in gear teeth. To derive this equation, refer to the adjucent figure, and the bending stress is;



$$\sigma = \frac{Mc}{I} = \frac{6 W_t \cdot L}{F \cdot t^2} \quad (1)$$

Where: $M = W_t \cdot L$

$c = t/2$

$I = F \cdot t^3/12$

Assume that the maximum stress in gear tooth occurs at point (a), so by similar triangles, we can write.

$$\frac{t/2}{x} = \frac{L}{t/2} \Rightarrow x = \frac{t^2}{4L} \quad (2)$$

By rearranging Eq.(1),

$$\sigma = \frac{6 W_t L}{F t^2} = \frac{W_t}{F} \cdot \frac{1}{t^2/6L} = \frac{W_t}{F} \cdot \frac{1}{t^2/4L} \cdot \frac{1}{4/6} \qquad (3)$$

By substituting the value of x from equ.(2) in equ.(3) and multiply the numerator and denominator by the circular pitch "P_c", we find

$$\sigma = \frac{W_t \cdot P_c}{F \cdot (\frac{2}{3}) X \cdot P_c} = \frac{W_t}{F \cdot (\frac{2}{3}) \frac{X}{P_c} P_c} = \frac{W_t}{F \cdot y \cdot P_c} \qquad (4)$$

where $y = \frac{2x}{3P_c}$ is the Lewis form factor

But most engineers prefer to employ the diametral pitch "P_d" in determining the stresses. This is done by substituting "$P_d = \pi / P_c$" and $Y = \pi . y$ in equ.(4), and this gives

$$\sigma = \frac{W_t \cdot P_d}{F \cdot Y} \qquad (5)$$

Where $Y = \pi \cdot y = \pi \cdot \frac{2X}{3P_c} = \frac{2X P_d}{3} = \frac{2X}{3m}$.

The use of this equation for "Y" means that only the bending of the tooth is considered and that the compression due to the radial component of the force is neglected. Values of "Y" obtained from this equation are tabulated in Table 14-2, Pg 718.

6. Dynamic Effects

When a pair of gears is driven at moderate or high speeds and noise is generated, it is certain that dynamic effect are presented. Then the stress equation; $\sigma = \frac{K_v \cdot W_t \cdot P_d}{F \cdot Y}$ or $\sigma = \frac{K_v \cdot W_t}{F \cdot m \cdot Y}$ (6)

Where 'K_v' is the velocity factor (dynamic factor) in term of the pitch-line velocity. Also, the face width "F" and the module "m" are in "mm".

$$K_v = \frac{3.05 + V}{3.05}$$ for cast iron, cast profile.

$$K_v = \frac{6.1 + V}{6.1}$$ for cut or milled profile.

$$K_v = \frac{3.56 + \sqrt{V}}{3.56}$$ for hobbed or shaped profile.

$$K_v = \sqrt{\frac{5.56 + \sqrt{V}}{5.56}}$$ for shaved or ground profile.

But if the gear have shaved teeth and ground teeth and there is no dynamic load, then $K_v = 1$. In the above equations (V) is in (m/s).

Moreover, the fatigue failure in gear teeth can address through expressing the fatigue stress-concentration factor K_f as:

$$K_f = H + \left(\frac{t}{r}\right)^{l} \left(\frac{t}{L}\right)^{M} \qquad (7)$$

Where $H = 0.34 - 0.4583662\,\phi$

$l = 0.316 - 0.4583662\,\phi$

$M = 0.290 + 0.4583662\,\phi$

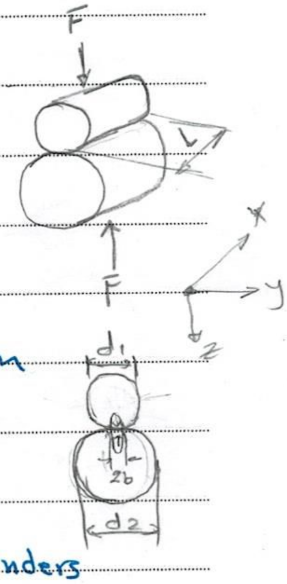
$r = (b - r_f)^2 / [(d/2) + b - r_f]$

In these equations L and t are from the layout in first figure in this chapter, ϕ is the pressure angle, r_f is the fillet radius, b is the dedendum, and d is the pitch diameter.

2. Surface Durability

The failure of the surfaces of gear teeth is generally called wear. Pitting can be defined as a surface fatigue failure due to many repetitions of high contact stresses. Other surface failure are scoring, which is a lubrication failure, and abrasion, which is wear due to the presence of foreign material.

Hertz theory can be used to obtain the surface-contact stress, which is shown the contact stress between two cylinders as shown in the equation below:



$$P_{max} = \frac{2F}{\pi b L} \qquad (8)$$

where: P_{max} = largest surface pressure

F = force pressing the two cylinders together

L = length of cylinder

b = half-width, which is obtained from

$$b = \left\{ \frac{2F}{\pi L} \frac{[(1-\nu_1^2)/E] + [(1-\nu_2^2)/E_2]}{(1/d_1) + (1/d_2)} \right\}^{1/2} \qquad (9)$$

where: ν_1 and ν_2 = Poisson's ratios of the two cylinders

E_1 and E_2 = Elastic modulus of the two cylinders

d_1 and d_2 = Diameters of the two contacting cyliders

By replacing "b" by $W^t/\cos\phi$, d by 2r, L by the face width F, and P_{max} by σ_c, the surface compressive stress (Hertzian stress) is found from the equation

$$\sigma_c^2 = \frac{W^t}{\pi . F . \cos\phi} \cdot \frac{(1/r_1) + (1/r_2)}{[(1-\nu_1^2)/E_1] + [(1-\nu_2^2)E_2]} \qquad (10)$$

Where r_1 and r_2 are the instantaneous values of radii of curvature on the pinion and gear tooth profiles at the contact point.

It is already noted that the wear occurs near the pitch line. The radii of curvature of the tooth profiles at the pitch are:

$$r_1 = \frac{d_P \sin\phi}{2} \quad \& \quad r_2 = \frac{d_G \sin\phi}{2} \qquad (11)$$

where: ϕ = the pressure angle.

d_P = the pitch diameter of the pinion.

d_G = the pitch diameter of the gear.

In order to simplify equation (10), an elastic coefficient "C_P" is defined by the equation

$$C_P = \left[\frac{1}{\pi \left[\frac{1-\nu_P^2}{E_P} + \frac{1-\nu_G^2}{E_G} \right]} \right]^{1/2} \qquad (12)$$

With this simplification, and the addition of a velocity factor k_v, equation (10) can be written as

$$\sigma_C = -C_P \left[\frac{k_v \cdot w^t}{F \cdot \cos\phi} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \right]^{1/2} \qquad (13)$$

Where the sign is negative because σ_C is a compressive stress.

3. AGMA stress Equations

Two fundamental stress equations are used in the AGMA methodology, one for bending stress and another for pitting resistance (contact stress). The fundamental equation for bending action is:

$$\sigma = W^t \cdot K_o \cdot K_v \cdot K_s \cdot \frac{1}{b\, m_t} \cdot \frac{K_H \cdot K_B}{Y_J} \qquad (14)$$

$$m_t = 1/P_d$$

Where: W^t = the tangential transmitted load (N)

K_o = the overload factor

K_v = the dynamic factor

K_s = the size factor

P_d = the transverse diameteral pitch.

b $\cancel{K_H}$ = the face width of the narrower member (mm)

K_H = the load-distribution factor

K_B = the rim-thickness factor

Y_J = the geometry factor for bending strength (which includes root fillet stress concentration factor K_f)

m_t = the transverse metric module.

The fundamental equation for pitting resistance (contact stress) is:

$$\sigma_c = Z_E \left(W^t K_o K_v K_s \frac{K_H}{d_{w1}\, b} \frac{Z_R}{Z_I} \right)^{1/2} \quad \text{(SI Units)} \qquad (15)$$

Where w^t, k_o, k_v, k_s, k_H and b are the same terms as defined above

Z_E = the elastic coefficient $(N/mm^2)^{1/2}$

Z_R = the surface Condition factor

d_{w1} = the pitch diameter of the pinion (mm)

Z_I = the geometry factor for pitting resistance

4. AGMA Strength Equations

Values for gear bending strength, designated here as "S_t", are to be found in Figs 14-2, 14-3, and 14-4, and in Tables 14-3 and 14-4. In this approach the strengths are modified by various factors that produce limiting values of the bending stress and the contact stress. The equation for the allowable bending stress is;

$$\sigma_{all} = \frac{S_t}{S_F} \frac{Y_N}{Y_\theta Y_z} \quad \text{(SI Units)} \qquad (16)$$

Where: S_t = the allowable bending stress (N/mm^2)

Y_N = the stress cycle factor for bending stress

Y_θ = the temperature factor

Y_z = the reliability factor

S_F = the AGMA factor of safty, a stress ratio

The equation for the allowable contact stress "$\sigma_{c,all}$" is;

$$\sigma_{c,all} = \frac{S_c}{S_H} \frac{Z_N Z_W}{Y_\theta Y_Z} \quad \text{(SI units)} \qquad (17)$$

Where: S_c = the allowable contact stress (N/mm^2)

Z_N = the stress cycle life factor

Z_W = the hardness ratio factors for pitting resistance

Y_θ = the temperature factors

Y_Z = the reliability factors

S_H = the AGMA factor of safty, a stress ratio

The values for the allowable contact stress, designated as 'S_c' are to be found in Fig. 14-5 and Tables 14-5, 14-6, and 14-7.

5. Geometry Factors I and J (Z_I and Y_J)

The determination of the geometry factors Z_I and Y_J depends upon the face-contact ratio "m_F", which is defined as:

$$m_F = \frac{F}{p_x} \qquad (18)$$

Where: p_x = the axial pitch

F = the face width

Note: For spur gears, $m_F = 0$ and conventional helical gears $m_F > 1$.

5.1 Bending-Strength Geometry Factor Y_J

The AGMA factor "Y_J" employs a modified value of the Lewis form factor, also denoted by "Y"; a fatigue stress-concentration factor "K_f"; and a tooth load-sharing ratio "m_N". The resulting equation for "Y_J" for spur and helical gears is

$$Y_J = \frac{Y}{K_f \cdot m_N} \qquad (19)$$

It is important to note that the form factor Y here is obtained from calculations with AGMA 908-B89, and is often based on the highest point of single-tooth contact.

The load-sharing ratio "m_N" is equal to the face width divided by the minimum total length of the lines of contact. This factor depends on the transverse contact ratio m_p, the face-contact ratio m_F, the effects of any profile modifications, and the tooth deflection.

For spur gears $\quad m_N = 1$

For helical gears having a face-contact ratio $m_F > 2$, the load-sharing ratio is given by the equation

$$m_N = \frac{P_N}{0.95 Z}$$

Where P_N is the normal base pitch and Z is the length of the line of action in the transverse plane (distance Lab in Fig. 13-15 pg 664).

In general, the value of the factor "Y_J" for spur gears with 20° pressure angle and full-depth teeth is found from Fig 14-6. While Figs 14-7 and 14-8 can be used for helical gears having 20° normal pressure angle and face-contact ratio of $m_F = 2$ or greater.

5.2 Surface-Strength Geometry Factor Z_I

The factor Z_I is also called the pitting-resistance geometry factor by AGMA. The factor valid for both spur and helical gears.

$$Z_I = \begin{cases} \dfrac{\cos\phi_t \cdot \sin\phi_t}{2 m_N} \cdot \dfrac{m_G}{m_G + 1} & \text{external gears} \\[2ex] \dfrac{\cos\phi_t \cdot \sin\phi_t}{2 m_N} \cdot \dfrac{m_G}{m_G - 1} & \text{internal gears} \end{cases} \qquad (20)$$

see Fig 13-22 (c) pg 672

Where: ϕ_t = the transverse pressure angle for helical gears or pressure angle for spur gears.

m_G = Speed ratio; $m_G = \dfrac{N_G}{N_P} = \dfrac{d_G}{d_P}$

$m_N = 1$ for spur gears

$m_N = \dfrac{P_N}{0.95 Z}$ for helical gears

P_N = the normal base pitch; $P_N = P_n \cos\phi_n$

P_n = the normal circular pitch

Z = the length of the line of action in the transverse plane.

$$Z = \left[(r_P + a)^2 - r_{bP}^2\right]^{1/2} + \left[(r_G + a)^2 - r_{bG}^2\right]^{1/2} - (r_P + r_G)\sin\phi_t \qquad (21)$$

Where: r_p and r_G = the pitch radii of pinion and gear

r_{bp} and r_{bG} = the base-circle radii of pinion and gear

a = the addendum

Note: (1) $r_b = r \cos \phi_t$

(2) Certain precaution must be taken in using Eq. (21); if any of the first two terms is larger than the third term, then it should be replaced by the third term.

6. The Elastic Coefficient C_p (Z_E)

Values of C_p may be computed directly from Eq. 12 or obtained from Table 14-8.

7. Dynamic Factor K_v

This factor is used to account for inaccuracies in the manufacture and meshing of gear teeth in action, which cause deviation from the uniform angular velocity of the gear pair.

AGMA uses transmission accuracy-level number "Q_v" to quantify gears into different classes according to manufacturing accuracy "tolerances".

$$K_v = \left(\frac{A + \sqrt{200V}}{A}\right)^B \qquad (V \text{ in m/s}) \qquad (22)$$

Where $A = 50 + 56(1 - B)$ and $B = 0.25(12 - Q_v)^{2/3}$ (23)

And the maximum velocity, representing the end point of the Q_v curve, is given by $(V_t)_{max} = [A + (Q_v - 3)]^2 / 200$ in m/s.

Also, figure 14-9 is a graph of K_v as a function of pitch-line speed for graphical estimates of K_v.

8. Overload Factor K_o

This factor is used to account for external load exceeding the normal tangential load W^t in a particular application. such as variations in torque due to firing of cylinders in internal combustion engines or reaction to torque variations in a piston pump drive. Figs. 14-17 and 14-18 show the values of K_o.

9. Surface Condition Factor C_f (Z_R)

This factor is used only in the pitting resistance equation, Eq. 15. It depends on surface finish, residual stress and plastic effect. Standard surface conditions for gear teeth have not yet been established, thus AGMA specifies a value of $C_f \geq 1$.

10. Size Factor K_s

The size factor reflects nonuniformity of material properties due to size. It depends upon tooth size, diameter of part, ratio of teeth size to diameter of part, face width, area of stress pattern, ratio of case depth to tooth size, and hardenability and heat treatment.

$$K_s = 1.92\left(\frac{F\sqrt{Y}}{P}\right)^{0.0535} \qquad (24)$$

If K_s in Eq. 24 is less than 1, use $K_s = 1$.

11. Load Distribution Factor K_H

The load distribution factor modified the stress equation to reflect nonuniform distribution of load across the line of contact. There are several reasons for nonuniform distribution of load such as; misalignment of the gear axis resulting from the deformation of the shaft carrying the gear, inaccuracy in manufacturing, and assembly. The following procedure is applicable to

- Net face width to pinion diameter ration $\frac{F}{d} \leq 2$
- Gear elements mounted between the bearings
- Face width up to 40 in "$F \leq 40$ in"
- Contact across the full width of the narrowest member.

The load distribution factor can be given by the face load distribution factor "C_{mf}", where

$$K_H = C_{mf} = 1 + C_{mc}(C_{pf} \cdot C_{pm} + C_{ma} \cdot C_e) \qquad (25)$$

Where:

$$C_{mc} = \begin{cases} 1 & \text{for uncrowned teeth} \\ 0.8 & \text{for crowned teeth} \end{cases}$$

$$C_{pf} = \begin{cases} F/10d - 0.025 & F \leq 1\text{ in} \\ F/10d - 0.0375 + 0.0125F & 1 < F \leq 17\text{ in} \\ F/10d - 0.1109 + 0.0207F - 0.000228F^2 & 17 < F \leq 40\text{ in} \end{cases}$$

Note: Values of $F/10d < 0.05$, $F/10d = 0.05$ is used.

$d = d_p$ "pitch diameter"

$$C_{pm} = \begin{cases} 1 & \text{for straddle-mounted pinion with } S_1/S < 0.175 \\ 1.1 & \text{for straddle-mounted pinion with } S_1/S \geq 0.175 \end{cases}$$

See Fig. 14-10 for definition of S and S_1. where S is the full span and S_1 is the distance from midspan.

$$C_{ma} = A + BF + CF^2$$

See table 14-9 for values of A, B, and C, Also C_{ma} can be found from Fig. 14.11.

$$C_e = \begin{cases} 0.8 & \text{for gearing adjusted at assembly, or compatibility is improved by lapping, or both} \\ 1 & \text{for all other Conditions} \end{cases}$$

12. Hardness Ratio Factor ~~Z_W~~ C_H

The pinion has less number of teeth than gear and consequently is subjected to more cycles of contact stress. Therefore, different treatment are used for the pinion to make it harder than gear. The hardness ratio factor is used for the gear only, and it's used to adjust the surface strengths for this effect.

$$C_H = 1 + A'(m_G - 1)$$

where $A' = 8.98 \times 10^{-3}\left(\frac{H_{BP}}{H_{BG}}\right) - 8.29 \times 10^{-3}$ $\quad 1.2 \le \frac{H_{BP}}{H_{BG}} \le 1.7$

H_{BP} = the Brinell hardness of the pinion

H_{BG} = the Brinell hardness of the gear

m_G = the speed ratio, $m_G = \frac{N_G}{N_P} = \frac{d_G}{d_P}$

See Fig. 14-12 for a graph of C_H.

For: $\frac{H_{BP}}{H_{BG}} < 1.2$, $A' = 0$

$\frac{H_{BP}}{H_{BG}} > 1.7$, $A' = 0.00698$

For surface hardened pinion with hardnesses of 48 Rockwell C scale (Rockwell C48) or harder run with through-hardened gear (180 -400 Brinell). The C_H factor is a function of pinion surface finish "f_p" and the mating gear hardness. Fig. 14-13 displays the relationships:

$$C_H = 1 + B'(450 - H_{BG})$$

Where: $B' = 0.00075 \exp[-0.0112 f_p]$

f_p = the surface finish of the pinion expressed as root-mean-square roughness R_a in μ in.

13. Stress Cycle Factors Y_N and Z_n

The AGMA strengths, bending "S_t" and contact "S_c", are based on 10^7 load cycles applied. The purpose of the load cycle factors Y_N and Z_n is to modify the gear strength for lives other than 10^7 cycles. Values for these factors are given in Figs. 14-14 and 14-15. Note that for 10^7 cycles $Y_N = Z_n = 1$.

14. Reliability Factor Y_Z (K_R)

The AGMA strengths S_t and S_c are based on 0.99 reliability. The values of Y_Z for reliabity other than 0.99 are found in table 14.10. For another reliability values, the Y_Z can be found

$$Y_Z = \begin{cases} 0.658 - 0.0759 \ln(1-R) & 0.5 < R < 0.99 \\ 0.5 - 0.109 \ln(1-R) & 0.99 \le R \le 0.9999 \end{cases}$$

15. Temperature Factor Y_θ (K_T)

For oil or gear-blank temperature up to 250°F (120°C), Use $Y_\theta = 1$. For higher temperature, Y_θ should be >1. Heat exchangers may be used to ensure that operating temperatures are considerably below this value.

16. Rim-Thickness Factor K_B

When the rim thickness is not sufficient to provide full support for the tooth root, the location of bending fatigue failure may be through the gear rim rather than at the tooth fillet. So, K_B is used to account for the increase in bending stress in thin-rimmed gears.

$$K_B = \begin{cases} 1.6 \ln\left(\frac{2.242}{m_B}\right) & m_B < 1.2 \\ 1 & m_B \geq 1.2 \end{cases}$$

Where m_B is the backup ratio and it can be found from

$$m_B = \frac{t_R}{h_t}$$

Where: t_R = rim thickness below the tooth

h_t = the tooth height

Also, K_B can be found from Fig. 14-16.

17. Safety Factor S_F and S_H

When designing gear sets, a factor of safety becomes a design factor (i.e. specified by the designer) indicating the desired strength to stress ratio.

- Bending stress factor of safety S_F is found as:

$$S_F = \frac{S_t \cdot Y_N / (K_T \cdot K_R)}{\sigma} = \frac{\text{Fully corrected bending strength}}{\text{bending stress}}$$

Where: σ is estimated from Eq. 14 and it's linearly related to the transmitted load "W^t"

$K_T = Y_\theta$ Temperature Factor

$K_R = Y_Z$ Reliability Factor

- Contact stress factor of safety S_H is found as:

$$S_H = \frac{S_c \cdot Z_N \cdot C_H / (K_T \cdot K_R)}{\sigma_c} = \frac{\text{Fully corrected contact strength}}{\text{contact stress}}$$

Where σ_c is estimated from Eq. 15 and it's not linear with the transmitted load "W^t".

C_H = the hardness-ratio factor and it is used only for the gear.

Because the difference in the relation of S_F and S_H with the transmitted load, a caution is required when comparing S_F with S_H in an analysis in order to ascertain the nature and severity of the threat of failure. Therefore

- Compare S_F with S_H^2 for linear or helical contact
- Compare S_F with S_H^3 for spherical contact (crowned teeth)

18. Analysis

Figs. 14-17 and 14-18 are a "road map" for bending fatigue and contact-stress fatigue, respectively. Fig. 14-17 identifies the bending stress equation, the endurance strength in bending equation, and the safety factor S_F. While Fig. 14-18 shows the contact-stress equation, the contact fatigue endurance strength equation, and the safety factor S_H.

Note:

- Most of the terms in the bending and contact stress or strength equations have the same values for both pinion and gear.
- The factors that could have two different values for the pinion and gear are: K_s, Y_J, K_B, S_t, S_c, Y_N, Z_n.

CHAPTER 16

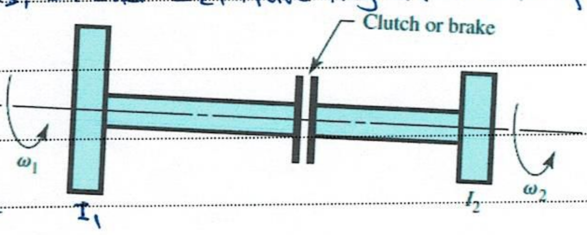
Clutches, Brakes, Couplings, and Flywheels

This chapter is concerned with a group of elements usually associated with rotation that have in common the function of storing and/or transferring rotational energy.

A clutch is a machine member used to connect a driving shaft to a driven shaft so that the driven shaft may be started or stopped at will, without stopping the driving shaft. The use of a clutch is mostly found in automobiles.

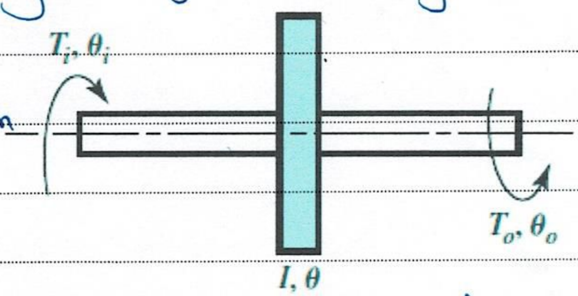
A brake is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine. In the process of performing this function, the brake absorbs either kinetic energy of the moving member or potential energy given up by objects being lowered by hoists, elevators, etc. The energy absorbed by brakes is dissipated in the form of heat.

A simplified dynamic representation of a friction clutch or brake is shown in Fig. below. Two inertias, I_1 and I_2, traveling at the respective angular velocities ω_1 and ω_2, one of which may be zero in the case of brakes, are to be brought to the same speed by engaging the clutch or brake. Slippage occurs because the two elements are running at different speeds and energy is dissipated during actuation, resulting in a temperature rise.



In analyzing the performance of these devises we shall be interested in the actuating force, the torque transmitted, the energy loss, and the temperature rise. The torque transmitted is related to the actuating force, the coefficient of friction, and the geometry of the clutch or brake. This is a problem in statics, which will have to be studied separately for each geometric configuration. However, temperature rise is related to energy loss and can be studied without regard to the type of brake or clutch, because the geometry of interest is that of the heat-dissipating surfaces.

A flywheel is an inertial energy-storage device. It absorbs mechanical energy by increasing its angular velocity and delivers energy by decreasing its velocity. Figure is a mathematical representation of flywheel. An input torque T_i, corresponding to a coordinate θ_i,

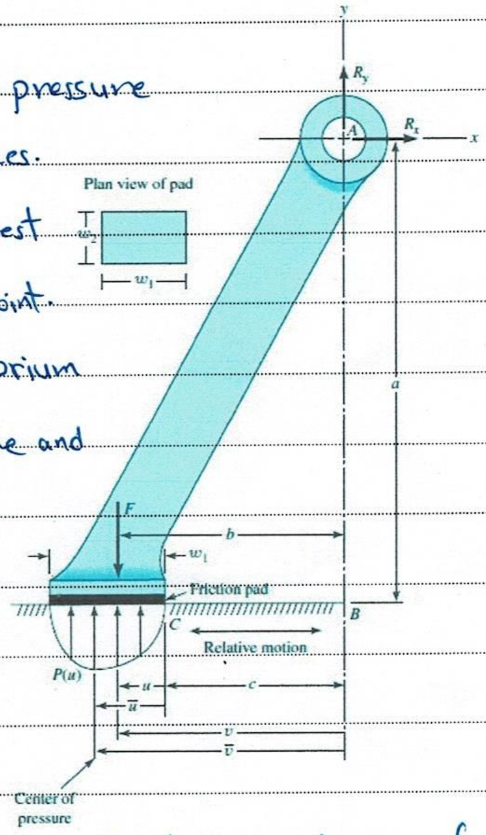


will cause the flywheel speed to increase. And a load or output T_o, with coordinate θ_o, will absorb energy from the flywheel and cause it to slow down. We shall be interested in designing flywheels so as to obtain a specified amount of speed regulation.

16.1 Static Analysis of Clutches and Brakes

The procedure that can be used to analyze many types of clutches and brakes are:

- Estimate, model, or measure the pressure distribution on the friction surfaces.
- Find a relationship between the largest pressure and the pressure at any point.
- Use the conditions of static equilibrium to find the braking force or torque and the support reactions.



Let us apply these tasks to the doorstop depicted in figure. The stop is hinged at point A. A normal pressure distribution $p(u)$ is shown under the friction pad as a function of position u, taken from the right edge of the pad. A similar distribution of shearing frictional traction is on the surface, of intensity $f\,p(u)$, in the direction of the motion of the floor relative to the pad, where f is the coefficient of friction. The width of the pad into the page is w_2. The net force in the y-direction and moment about C from the pressure are respectively,

$$N = w_2 \int_0^{w_1} p(u)\,du = p_{av}\, w_1 w_2$$

$$w_2 \int_0^{w_1} p(u)\,u\,du = \bar{u}\, w_2 \int_0^{w_1} p(u)\,du = p_{av}\, w_1 w_2 \bar{u}$$

$$\overset{\rightarrow}{\sum} F_x = R_x \mp w_2 \int_0^{w_1} f\, p(u)\, du = 0$$

$$R_x = \pm f\, w_1 w_2\, p_{av}$$

where − or + is for rightward or leftward relative motion of the floor, respectively. f is constant.

$$\uparrow \sum F_y = -F + w_2 \int_0^{w_1} p(u)\, du + R_y = 0$$

$$R_y = F - p_{av}\, w_1 w_2$$

The moments about the pin located at A are

$$\sum M_A = F b - w_2 \int_0^{w_1} p(u)\,(c+u)\, du \mp a f w_2 \int_0^{w_1} p(u)\, du = 0$$

$$F = \frac{w_2}{b}\left[\int_0^{w_1} p(u)\,(c+u)\, du \pm a f \int_0^{w_1} p(u)\, du\right]$$

Can F be equal to or less than zero? only during rightward motion of the floor is equal to or less than zero.

$$\Rightarrow \int_0^{w_1} p(u)\,(c+u)\, du - a f \int_0^{w_1} p(u)\, du \leq 0$$

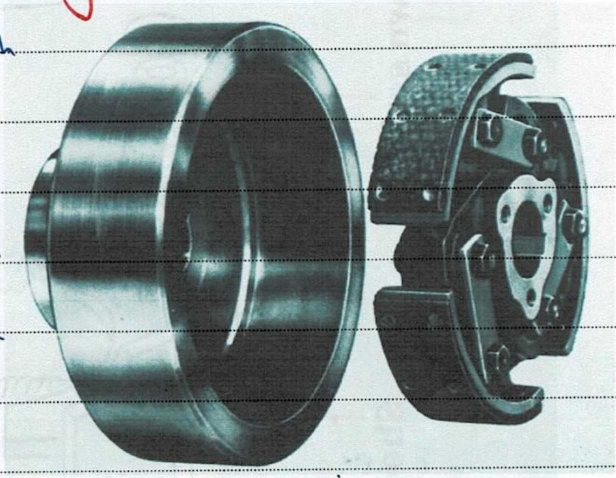
$$f_{cr} \geqslant \frac{1}{a}\, \frac{\int_0^{w_1} p(u)\,(c+u)\, du}{\int_0^{w_1} p(u)\, du} = \frac{1}{a}\, \frac{c \int_0^{w_1} p(u)\, du + \int_0^{w_1} p(u)\, u\, du}{\int_0^{w_1} p(u)\, du}$$

$$f_{cr} \geqslant \frac{c + \bar{u}}{a}$$

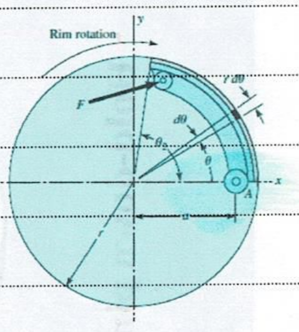
where f_{cr} = coefficient of friction
$\bar{u}$ = the distance of the center of pressure from the right edge of the pad.

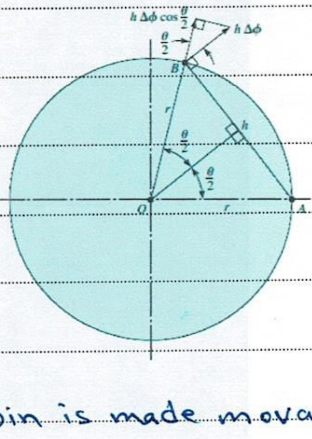
16.2 Internal Expanding Rim Clutches and Brakes

The internal-shoe rim clutch shown in figure consists essentially of three elements: the mating frictional surface, the means of transmitting the torque to and from the surfaces, and the actuating mechanism.



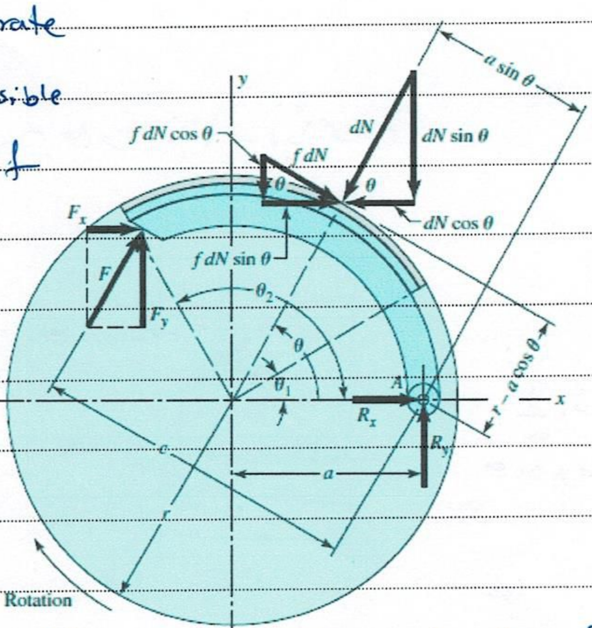
To analyze an internal-shoe device, refer to figure, which shows a shoe pivoted at point A, with the actuating force acting at the other end of the shoe. Since the shoe is long, we cannot make the assumption that the distribution of normal forces is uniform. The mechanical arrangement permits no pressure to be applied at the heel, and we will therefore assume the pressure at this point to be zero.





It is the usual practice to omit the friction material for a short distance away from the heel (point A). In some designs the hinge pin is made movable to provide additional heel pressure.

A good design would concentrate as much frictional material as possible in the neighborhood of the point of maximum pressure, as shown in figure. In this figure the frictional material begins at an angle θ_1, measured from the hinge pin A, and ends at an angle θ_2.



The hinge-pin reactions are R_x and R_y. The actuating force F has components F_x and F_y and operates at distance "c" from the hinge pin. At any angle θ from the hinge pin there acts a differential normal force dN whose magnitude is

$$dN = p\,b\,r\,d\theta$$

Where b is the face width of the frictional material. The relationship between the pressure p acting upon an element of area of the frictional material located at an angle θ from the hinge pin with the maximum pressure p_a located at an angle θ_a from the hinge pin is

$$p = \frac{p_a}{\sin\theta_a} \sin\theta$$

$$\therefore \quad dN = \frac{p_a\, b\, r \sin\theta \, d\theta}{\sin\theta_a}$$

Wher dN = the normal force

The normal force dN has two components, horizontal "$dN\cos\theta$" and vertical "$dN\sin\theta$". The frictional force $f dN$ has horizontal and vertical components whose magnitudes are $f dN \sin\theta$ and $f dN \cos\theta$, respectively.

$$M_f = \int f\, dN (r - a\cos\theta) = \frac{f p_a b r}{\sin\theta_a} \int_{\theta_1}^{\theta_2} \sin\theta (r - a\cos\theta)\, d\theta$$

$$M_N = \int dN (a\sin\theta) = \frac{p_a b r a}{\sin\theta_a} \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta$$

Where: M_f = the moment of frictional forces.

M_N = the moment of normal forces.

The actuating force F must balance these moments, thus

$$F = \frac{M_N - M_f}{C}$$ Clockwise rotation

When $M_N = M_f$ ⇒ self-locking is obtained, and no actuating force is required. This furnishes us with a method for obtaining the dimensions for some self-energizing action. Thus the dimension "a" in the fig must be such that

$$M_N > M_f$$

The torque T applied to the drum by the brake shoe is the sum of the frictional forces $f\,dN$ times the radius of the drum

$$T = \int f r\, dN = \frac{f p_a b r^2}{\sin\theta_a} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = \frac{f p_a b r^2 (\cos\theta_1 - \cos\theta_2)}{\sin\theta_a}$$

The hinge-pin reactions are found by taking a summation of the horizontal and vertical forces. Thus.

$$R_x = \int dN \cos\theta - \int f\, dN \sin\theta - F_x$$

$$= \frac{p_a b r}{\sin\theta_a}\left(\int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta - f \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta\right) - F_x$$

$$R_y = \int dN \sin\theta + \int f\, dN \cos\theta - F_y$$

$$= \frac{p_a b r}{\sin\theta_a}\left(\int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta + f \int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta\right) - F_y$$

The direction of the frictional forces is reversed if the rotation is reversed. Thus, for counterclockwise rotation the actuating force is

$$F = \frac{M_N + M_f}{c}$$

$$R_x = \frac{p_a b r}{\sin\theta_a}\left(\int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta + f \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta\right) - F_x$$

$$R_y = \frac{p_a b r}{\sin\theta_a}\left(\int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta - f \int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta\right) - F_y$$

Let assume the following to simplify the equations

$$A = \int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta = \left(\frac{1}{2} \sin^2\theta\right)_{\theta_1}^{\theta_2}$$

$$B = \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta = \left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)_{\theta_1}^{\theta_2}$$

Then, for clockwise rotation, the hinge-pin reactions are:

$$R_x = \frac{P_a b r}{\sin \theta_a}(A - fB) - F_x$$

$$R_y = \frac{P_a b r}{\sin \theta_a}(B + fA) - F_y$$

For counterclockwise rotation,

$$R_x = \frac{P_a b r}{\sin \theta_a}(A + fB) - F_x$$

$$R_y = \frac{P_a b r}{\sin \theta_a}(B - fA) - F_y$$

In using these equations, the reference system always has its origin at the center of the drum. The positive x-axis is taken through the hinge pin. The positive y-axis is always in the direction of the shoe, even if this should result in a *left-handed* system. See the assumptions in the book.